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CONTENTS

PART I

GENERAL ASYMMETRICAL NETWORKS

PART II

ROTATING MACHINERY

TO

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PREFACE

This volume contains a series of lectures delivered to students in the Advanced Course in Engineering of the General Electric Company. The subject matter represents a short outline of the tensorial method of attack of certain electrical-engineering problems that has appeared in three more exhaustive publications* and in several shorter papers since 1932. Although no additional *basic* concepts are introduced that have not appeared in the other publications, still several old topics are presented from a new point of view and other new subjects are touched upon, such as mercury-arc rectifier circuits. Among the new groups of transformations introduced are those establishing equivalent circuits for rotating machines (such as the capacitor motor) that can be set up on the a-c. network analyzer.

The subject matter has been selected from the point of view of the power engineer and is divided into two parts. The first part deals with the invariant theory of general asymmetrical networks, without inquiring too closely what the individual "coil" and its impedance Z stand for. The network may be stationary or rotating; the performance may be transient or steady-state. The second part undertakes a more detailed analysis of one special type of asymmetrical network, namely, rotating machines.

The purpose of this volume is to develop a new method of reasoning in analyzing engineering problems and not to study in detail any particular structure. There are no speed-torque curves or descriptions of performances of systems. The volume is restricted to the presentation of a unified method of analysis. Although the method of reasoning is

* Kron, "Tensor Analysis of Networks," John Wiley & Sons, January, 1939.

Kron, "The Application of Tensors to the Analysis of Rotating Electrical Machinery," Parts I–XVI, *General Electric Review*, May, 1938.

Kron, "The Application of Tensors to the Analysis of Rotating Electrical Machinery." A series of articles that appeared in the *General Electric Review* beginning April, 1935, of which Parts XVII and XVIII (May and October, 1938) are not included in the bound volume.

These three publications will be referred to throughout the text as *T.A.N.*, *A.T.E.M.*, and *G.E.R.*, respectively.

new, the results arrived at are in a form used by engineers with whom the author is in contact.

At first reading the following chapters may be considered: 1–4, 6, 12, 13, 15–18, 21–23, 28, 30, 32.

GABRIEL KRON

SCHENECTADY, N. Y.
February 9, 1942

INTRODUCTION

One of the purposes of this and the other books of the author is to establish, manipulate, and solve the equations of performance of complex engineering systems in an *organized* manner instead of haphazardly and to utilize this organization to obtain new information about the systems. In the following, only the *setting up* of equations will be studied in detail, and of the many manipulations only one process—the elimination of variables—will be introduced. Since practically all the differential equations introduced can be solved, if at all, by well-known methods, the systematic solution of systems of differential equations is not undertaken in these pages. In textbooks on matrices the reader will find a wealth of material on the systematic solution of sets of differential equations.

The organization is undertaken with the aid of a mathematical tool known as tensor analysis, which has been found to be the natural tool for investigating phenomena taking place in the actual physical world or in the abstract spaces invented by human imagination. However, the manner of application of these modern concepts for the problems of the engineer differs radically from the point of view adopted by the physicist (or the geometer). This radical departure is necessitated by the different goal aimed at by the two groups of specialists.

Tensor analysis has hitherto been used exclusively to establish the invariant laws of nature in the form of tensor equations that are independent of the reference frame employed. Very little attention has been paid, however, to expanding these symbolic equations to particular cases. In these volumes, on the contrary, the establishment of symbolic equations is only a stepping-stone toward the final goal of constructing a smooth-running mechanism that automatically unfolds the relatively few symbolic equations to apply to the infinite variety of *specific* problems with which an industrial civilization confronts the engineer.

This mechanism is nothing more than a method of reasoning, a philosophy, that serves as a pathfinder while the engineer cuts his way across the labyrinth of interrelated phenomena. A short outline of the proposed method of attack on engineering problems (whether they are electrical or mechanical phenomena) is given here.

Let the transient and steady-state performance of an engineering structure, say a turbine-governing system or an electric speed drive, be determined. The steps are as follows:

1. Do not analyze the *given* system immediately, since it is complicated. Instead, first set up the equations of *another* related system which is much simpler to analyze (or whose equations have already been established on a previous occasion).
2. Then change the equations of the simpler system to those of the complex system by a routine procedure.

Tensor analysis supplies the routine rules by which the equations of the simpler (or known) system are changed to those of the given system.

The question immediately arises: How are the simpler systems established? Two procedures are available, to be used independently or simultaneously.

1. *Break up the complex system into several component systems by removing certain strategically located interconnections so that each component should be easy to analyze. This break-up may be accomplished in several successive steps.*

For a turbine-governing system, say, the system is divided into the governor, the linkage, the pilot valve, and the turbine, and the performance of each is studied as if the others were not present. For an electric speed drive, the system is divided into the synchronous motor, the induction motor, and the stationary network.

Now, if the equations of each of these component systems have not been established before, then each component is again subdivided into still smaller components whose equations are easy to establish.

The collection of component systems, which forms the *last step* in the necessary subdivision, will be called the "primitive system."

Once the equation of a component part (say, the governor) has been established, there is no more necessity to establish its equation all over again when it is used as a component part of a different engineering system. That is, the results of all investigations in the language of tensors may be stored away for future use in different types of problems just as standardized machine parts are stored away to be reassembled in a variety of structures.

2. In addition to breaking up the complex system into several component systems, *assume new, simpler types of reference frames either in the original or in the broken-up systems.*

For instance, instead of curvilinear axes, assume rectilinear axes if

possible; or, instead of brushes at an angle, assume brushes along the main poles; etc. The new axes may be actually existing or hypothetical axes (like symmetrical components or normal coordinates, for example).

The routine procedure of going from the equations of the "primitive system" to the equations of the actual system is usually referred to as "transformation theory" or "transformation of reference frames." This process is the backbone of tensor analysis.

It is surprising how few ultimate types of elements there are that form the building blocks of the great variety of engineering structures. Most stationary networks consist of a collection of one-dimensional "coils" only; all rotating machines consist only of a collection of two-dimensional "windings." The great variety of structures differ only by the manner of interconnections of these ultimate coils and windings, and the variety of theories differ only by the type of hypothetical reference frame assumed.

It is only the study of the ultimate building blocks that requires analytical work. The interconnection of these units into the given system is a routine procedure.

Of course, many ideas of tensor analysis have been and are utilized by engineers in their daily work without using the word "tensor." The present study undertakes a systematization and extension of those loose or half-baked ideas and "hunches."

Although the method of reasoning will be employed here only for stationary and rotating electrical networks, *exactly the same* reasoning applies also to mechanical and other physical systems. That is, *all reasonings and all symbolic formulas to be studied are independent of electrical engineering.* The electrical applications are only illustrations.

It should be mentioned that only the *second* step of changing the reference frame on a given system has been used by geometers in differential geometry by employing the apparatus of tensor analysis. However, the first step of tearing a structure into several component parts, or rather transforming the equations of different structures into each other, has not been employed as yet in geometry. It is this very process of building up the equations of complex physical structures from those of their component parts that serves as the key to the tensorial analysis of engineering structures. Without this process every individual machine and system presents an isolated problem to be analyzed anew from the very fundamentals.

Only during the last few years has a similar study been undertaken

by geometers in topology, using reasonings, concepts, and the apparatus of tensor analysis analogous to those employed by electrical engineers.* It is rather interesting that Kirchhoff laid the foundation of topology by his study of electrical networks. It is not a coincidence but a consequence of some hitherto hidden relation between the properties of space and those of electricity that the science of electrical engineering and that of topology meet again on a common ground when both are viewed from an invariant point of view.

As first steps to the type of organization undertaken by the method of tensors, the use of matrices and three-dimensional vectors, both familiar concepts to engineers, may be considered. Matrices have been extensively employed by mathematicians in function-theoretical investigations, for instance in determining the characteristics of the roots of differential equations. Frazier, Duncan, and Collar † have used matrices in investigating the differential equations that arise in mechanical vibration problems. Feldtkeller,‡ also Strecker, Cauer, and their followers, have used matrices in synthesis problems of four-terminal communication networks.

The vectors of conventional vector analysis (also the dyadics, triadics, and, in general, the "polyadics") form a type of organization different from matrices, though matrices of various dimensions always arise whenever vectors are represented along some particular reference frame. Whereas matrices owe their existence to arbitrary mathematical definitions, vectors have an independent physical existence of their own, and their mathematical definitions try only to embody the physical characteristics of vectors endowed by nature. That is, *the concepts of conventional vector analysis are matrices come to life.*

Now *tensors may be looked upon as vectors* (or rather polyadics) *come of age.* While vectors can represent only three variables simultaneously, tensors may be used in problems with any number of variables. The use of conventional vectors is restricted to special types of reference frames drawn in special types of spaces. No such limitations are imposed upon tensors.

That is, "tensor" is just another name for "physical entity." *Tensor analysis is the study of physical phenomena in terms of the physical entities themselves.* It also supplies a routine mechanism to express the behavior of these entities in a mathematical form along any desired reference frame.

* Tucker, "Discussion on Tensor Analysis," *Electrical Engineering*, 1937, p. 619.

† "Elementary Matrices," Oxford University Press, 1938.

‡ "Fernmeldtechnik," Springer, Berlin, 1938.

Since engineers deal with more complicated and interrelated physical phenomena than physicists or geometers, tensor analysis is an engineering tool par excellence, and it might have been invented and developed by engineers had not engineering often been restricted in past decades to a cut-and-try art. As analytical methods come into more prominence and the complexity of engineering problems increases, the need of putting to practical use the organizing ability of tensorial methods will become more pressing.

Examples of such pressing needs are the equivalent circuits of single and interconnected rotating machines. The computations of the performance of modern interconnected systems are so long and time-consuming that the aid of calculating machines, such as the a-c. network analyzer, must be resorted to. The establishment of equivalent circuits requires just the type of organized steps and physical pictures that the tensorial method of attack, presented in this book, supplies automatically.

Other examples are the stability and hunting studies of engineering structures that are today in the foreground of attention because of the increased use of automatic control devices. It is well known that the conventional application of the Lagrangean equations to the study of small oscillations—as given in textbooks on dynamics or on electrical machinery—do not lead to tensor (invariant) equations. As a consequence the resulting equations do not give a *complete* physical picture (except in special cases) of what actually takes place in the system during small oscillations, even though the equations do give correct numerical answers. This lack of completeness shows up in any attempt to visualize the phenomena of hunting or in attempts to construct physical models.*

To establish the invariant form of hunting equations and thereby to express the phenomena of small oscillations in terms of measurable and visualizable physical quantities, it is necessary to employ such advanced concepts of tensor analysis as the Riemann-Christoffel curvature tensor (discovered first by Riemann about a century ago).

It is the avowed purpose of these volumes to introduce into the study of engineering structures only such concepts as physicists have developed for the study of the simplest unit of the structure. Every effort has been made at the same time to introduce only the absolute minimum of concepts into engineering and only those that form the very foundation of theoretical physics. The formulas and methods

* Kron, "Equivalent Circuits for the Hunting of Electrical Machinery," *Trans. A.I.E.E.*, 1942.

of attack proposed are based upon the conviction that the science of engineering differs from the science of physics only in:

1. Using a larger number of variables.
2. Employing greater variety of reference frames.
3. Constructing more complex spaces.

But the basic symbols used in both sciences are identical; they *must be identical* by virtue of the very identity of the physical phenomena dealt with.

PART I

GENERAL ASYMMETRICAL NETWORKS

CHAPTER 1

THE ALGEBRA OF *N*-WAY MATRICES *

N-WAY MATRICES

The presentation of tensor analysis is facilitated by an acquaintance with the algebra of matrices.

A set of quantities may be arranged in various dimensions and denoted by a single symbol. Such a set is a row

$$\mathbf{i} = \begin{array}{|c|c|c|c|c|c|}\hline 3 & 2 & -5 & 4 & 7 & 0 \\ \hline \end{array} = 1 - \text{matrix}$$

a rectangle,

$$\mathbf{A} = \begin{array}{|c|c|c|}\hline 1 & 3 & 7 \\ \hline 0 & -2 & 9 \\ \hline 5 & x & x^2 \\ \hline \end{array} \qquad \mathbf{B} = \begin{array}{|c|c|}\hline \cos\alpha & x \\ \hline 23 & -7x \\ \hline y & -y^2 \\ \hline \end{array} = 2 - \text{matrix}$$

or a cube, Fig. 1.1.

In general such multi-dimensional sets are called "*n*-way matrices" or "*n*-matrices" or briefly "matrices." (A "2-matrix" is often called a

A= A=

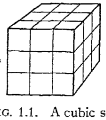


FIG. 1.1. A cubic set.

"matrix" when no misunderstanding may arise. A 1-matrix is also called a "linear matrix" or "column matrix.") Each number is called an "element."

A single quantity like 5 or x^2 may be called a "0-matrix" (zero-dimensional matrix).

* *T.A.N.*, Chapters I and II.

The number of rows and columns and layers may vary from one to infinity, depending on the problem. The theory of n-matrices with infinite number of rows will not be considered here.

In print, matrices are denoted by bold-face letters as above or by brackets as $\mathbf{A} = [A]$. In writing, matrices are usually represented by a bar over the letter.

EXAMPLES OF N-WAY MATRICES

Let a stationary network with four meshes be given. Then (Fig. 1.2):

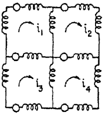


FIG. 1.2. A four-mesh network.

1. The four mesh currents may be arranged in a 1-matrix and denoted by one symbol as

$$\mathbf{i} = \begin{array}{|c|c|c|c|}\hline i_1 & i_2 & i_3 & i_4 \\ \hline \end{array}$$

2. Similarly the impressed voltages around the meshes form a 1-matrix

$$e = \begin{array}{|c|c|c|c|}\hline e_1 & e_2 & e_3 & e_4 \\ \hline \end{array}$$

Each of the components may be d-c or a-c or instantaneous or a Heaviside unit function, etc.

3. The self and mutual impedances of the meshes may be arranged as a 2-matrix

$$\mathbf{Z} = \begin{array}{|c|c|c|c|}\hline Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ \hline Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ \hline Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ \hline Z_{41} & Z_{42} & Z_{43} & Z_{44} \\ \hline \end{array}$$

where

$$Z_{11} = R_{11} + L_{11}p + 1/pC_{11} + Lp\theta$$

or

$$Z_{11} = R_{11} + jX_{11} + Xv$$

Each component may be a real or complex number or may contain the differential operator $p = d/dt$.

4. The instantaneous power input, or stored magnetic energy, or electrostatic energy in the whole system, is a single number (or single function of time) and each is a 0-matrix.

ADDITION OF N-MATRICES

N-matrices may be manipulated as ordinary quantities with certain precautions. Only 0-, 1-, and 2-matrices, and their addition, multiplication, and division, will be considered here.

Only n-matrices of the same dimensions and the same number of rows may be added. They are added by adding corresponding components.

$$\mathbf{i} = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 5 & -4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$$

$$\mathbf{e} = \begin{bmatrix} 0 & -2 & 1 \end{bmatrix} \tag{1.1}$$

$$\mathbf{i} + \mathbf{e} = \begin{bmatrix} 1 & 1 & 6 \end{bmatrix} \qquad \mathbf{A} + \mathbf{B} = \begin{bmatrix} 3 & 2 \\ 4 & -2 \end{bmatrix}$$

Multiplication of N-Matrices

1. 0-Matrix and n-Matrix $a\,\mathbf{B}$. Any n-matrix is multiplied by a single number (0-matrix) by multiplying each element by the given number.

$$a\mathbf{A} = 2 \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & -1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & -2 \\ 0 & 4 & 2 \end{bmatrix} \tag{1.2}$$

2. Two 1-Matrices $\mathbf{i}\cdot\mathbf{e}$. Multiply corresponding elements and add them. The product is a single number, a 0-matrix. The product is usually denoted by a dot.

$$\mathbf{i} = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \qquad \mathbf{i}\cdot\mathbf{e} = (1)(0) + (3)(-2) + (5)(1)$$

$$\mathbf{e} = \begin{bmatrix} 0 & -2 & 1 \end{bmatrix} \qquad = 0 - 6 + 5 = -1 \tag{1.3}$$

3. 2-Matrix and a 1-Matrix $\mathbf{A}\cdot\mathbf{i}$. Draw a horizontal and a vertical arrow as shown, and multiply each row of the 2-matrix by the 1-matrix,

giving a single number. These single numbers are arranged in a 1-matrix in their proper order.

$$\mathbf{A}\cdot\mathbf{i} = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 5 & 6 \\ 1 & 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3\cdot 1 + 1\cdot 2 + 4\cdot 3 \\ 2\cdot 1 + 5\cdot 2 + 6\cdot 3 \\ 1\cdot 1 + 2\cdot 2 + 5\cdot 3 \end{bmatrix} = \begin{bmatrix} 17 \\ 30 \\ 20 \end{bmatrix} \tag{1.4}$$

4. *1-Matrix and a 2-Matrix* **i·A**. Again draw first a horizontal, then a vertical arrow, and multiply as above. The result again is a 1-matrix.

$$\mathbf{i}\cdot\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 & 4 \\ 2 & 5 & 6 \\ 1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 10 & 17 & 31 \end{bmatrix} \tag{1.5}$$

5. *Two 2-Matrices* **A·B**. Again draw first a horizontal, then a vertical arrow, and multiply each row of the first by each column of the second 2-matrix (as the arrow indicates). Each product is placed in the corresponding place of the resultant matrix as shown.

$$\mathbf{A}\cdot\mathbf{B} = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 5 & 6 \\ 1 & 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 & 4 \\ 2 & 5 & 6 \\ 1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & 1\cdot 1 + 2\cdot 5 + 5\cdot 2 & \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & 21 & \end{bmatrix} \tag{1.6}$$

DIVISION WITH *N*-MATRICES

An n-matrix can be divided only by a single number (0-matrix) and a square 2-matrix.

1. An n-matrix is divided by a single number by dividing each of its elements by the number.

$$\mathbf{Z}/2 = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & -8 \\ 0 & 2 & 5 \end{bmatrix} \div 2 = \begin{bmatrix} 3/2 & 2 & 5/2 \\ 3 & 7/2 & -4 \\ 0 & 1 & 5/2 \end{bmatrix} \tag{1.7}$$

2. Division by a 2-matrix **Z** is represented by a multiplication with

its inverse $\mathbf{Z}^{-1}$. Finding the inverse of $\mathbf{Z}$ is analogous to solving a set of simultaneous equations by Cramer's rule.

In order to find the *inverse* of a matrix $\mathbf{Z}$, one first has to know how to find the (*a*) "determinant" of a matrix, (*b*) "minor" of an element of a matrix.

(*a*) The *determinant* of $\begin{vmatrix} A & B \\ C & D \end{vmatrix}$ is $AD - CB$ = a number $\quad$ 1.8

The *determinant* of $\begin{vmatrix} A & B & C \\ D & E & F \\ G & H & K \end{vmatrix}$ is $AEK + BFG + DHC - GEC - DBK - AHF$ = a number $\quad$ 1.9

(*b*) The *minor* of an element is found by cancelling the row and column of the matrix passing through the element and calculating the determinant of the remaining matrix. For instance, the minor of B in the last matrix is $DK - GF$.

Inverse Calculation of Z

The inverse of $\mathbf{Z}$ is found in *four* steps:

1. Interchange rows and columns (i.e., find $\mathbf{Z}_t$).
2. Replace each element by its minor.
3. Divide each element by the determinant of $\mathbf{Z}$.
4. Multiply the elements alternately by plus or minus 1 according to the scheme of Fig. 1.3.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

FIG. 1.3.

As an example let the inverse of

$$\mathbf{Z} = \begin{vmatrix} A & B & C \\ D & E & F \\ G & H & K \end{vmatrix}$$

be calculated

1. The transpose of $\mathbf{Z}$ is

$\mathbf{Z}_t =$

A	D	G
B	E	H
C	F	K

2. The minor of each element is

$EK-FH$	$BK-HC$	$BF-EC$
$DK-GF$	$AK-GC$	$AF-DC$
$DH-GE$	$AH-GB$	$AE-DB$

3. Divide each element by

$$\text{Det} = AEK + BFG + DHC - GEC - DBK - AHF$$

4. Multiply each element alternately by plus or minus 1

$(EK-FH)/\text{Det}$	$(HC-BK)/\text{Det}$	$(BF-EC)/\text{Det}$
$(GF-DK)/\text{Det}$	$(AK-GC)/\text{Det}$	$(DC-AF)/\text{Det}$
$(DH-GE)/\text{Det}$	$(GB-AH)/\text{Det}$	$(AE-DB)/\text{Det}$

1.10

The fraction $1/\text{Det}$ may be written outside the matrix as a factor.

IMPORTANT 2-MATRICES

1. When the rows and columns of a matrix $\mathbf{A}$ are interchanged, the resultant matrix is called the "transposed" matrix and is denoted as $\mathbf{A}_t$. For example,

$\mathbf{A} =$

1	2	3
4	5	6

$\mathbf{A}_t =$

1	4
2	5
3	6

1.11

The transpose of a transposed matrix is the original

$$(\mathbf{A}_t)_t = \mathbf{A} \qquad 1.12$$

2. The "unit matrix" has unity in its main diagonal and zero elsewhere.

$$\mathbf{1} = \begin{array}{|c|c|c|} \hline 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \hline \end{array} \tag{1.13}$$

Any matrix multiplied by **1** is unchanged.

$$\mathbf{1}\cdot\mathbf{A} = \mathbf{A}\cdot\mathbf{1} = \mathbf{A} \tag{1.14}$$

The unit matrix is used in factoring.

$$\mathbf{A} + \mathbf{A}\cdot\mathbf{B} = \mathbf{A}\cdot(\mathbf{1} + \mathbf{B}) \tag{1.15}$$

3. The "zero matrix" has zero for all its components.

$$\mathbf{0} = \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 0 & 0 \\ \hline \end{array} \tag{1.16}$$

Any matrix multiplied by **0** becomes zero.

$$\mathbf{0}\cdot\mathbf{A} = \mathbf{0} \tag{1.17}$$

4. A "diagonal matrix" has components only along the main diagonal.

$$\mathbf{Z} = \begin{array}{|c|c|c|} \hline a & & \\ \hline & b & \\ \hline & & c \\ \hline \end{array} \tag{1.18}$$

5. A "symmetrical matrix" is symmetrical with respect to the main diagonal line.

$$\mathbf{A} = \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 4 & 2 & 5 \\ \hline 6 & 5 & 3 \\ \hline \end{array} \tag{1.19}$$

For a symmetrical 2-matrix $\mathbf{A}$

$$\mathbf{A} = \mathbf{A}_t \qquad 1.20$$

6. A "skew-symmetric" matrix has components with opposite signs on the two sides of the main diagonal.

$$\mathbf{A} = \begin{array}{|c|c|c|} \hline 0 & 2 & 3 \\ \hline -2 & 0 & 4 \\ \hline -3 & -4 & 0 \\ \hline \end{array} \qquad 1.21$$

7. Any matrix $\mathbf{A}$ may be divided into the sum of a symmetrical matrix $\mathbf{B}$ and a skew-symmetric matrix $\mathbf{C}$. That is, $\mathbf{A} = \mathbf{B} + \mathbf{C}$, where

$$\mathbf{B} = \frac{\mathbf{A} + \mathbf{A}_t}{2} \quad \text{and} \quad \mathbf{C} = \frac{\mathbf{A} - \mathbf{A}_t}{2} \qquad 1.22$$

FORMS

(*a*) The product of two 1-matrices $\mathbf{e}$ and $\mathbf{i}$ as $\mathbf{e}\cdot\mathbf{i}$ (a 0-matrix) is called a "linear form." If $\mathbf{i}$ represents the mesh currents of a network and $\mathbf{e}$ the impressed voltages, then $\mathbf{e}\cdot\mathbf{i} = P$ is the "power input," a linear form.

The double product of a 1-matrix $\mathbf{i}$ and a 2-matrix $\mathbf{L}$ as $\mathbf{i}\cdot\mathbf{L}\cdot\mathbf{i}$ (a 0-matrix) is called a "quadratic form." If $\mathbf{i}$ represent the mesh currents and $\mathbf{L}$ the self and mutual inductances of a network, then $(\frac{1}{2})$ $\mathbf{i}\cdot\mathbf{L}\cdot\mathbf{i}$ represents the instantaneous stored magnetic energy in the system.

(*b*) One property of quadratic forms should be mentioned

$$\mathbf{i}\cdot\mathbf{L}\cdot\mathbf{i} = \mathbf{i}\cdot\left(\frac{\mathbf{L} + \mathbf{L}_t}{2}\right)\cdot\mathbf{i} \qquad 1.23$$

That is, *the 2-matrix of a quadratic form is always symmetrical* (or rather the skew-symmetric part of $\mathbf{L}$, namely $(\mathbf{L} - \mathbf{L}_t)/2$, multiplied by $\mathbf{i}$ twice always gives zero).

PROPERTIES OF THE INVERSE MATRIX $\mathbf{Z}^{-1}$

1. Only a square matrix has an inverse.
2. $\mathbf{Z}^{-1}$ is also a square matrix.
3. If $\mathbf{Z}$ is a symmetrical matrix, $\mathbf{Z}^{-1}$ is also symmetrical.
4. If $\mathbf{Z}$ is a diagonal matrix, then $\mathbf{Z}^{-1}$ is also diagonal.

$$\mathbf{Z} = \begin{array}{|c|c|c|c|}\hline a & & & \\ \hline & b & & \\ \hline & & c & \\ \hline & & & d \\ \hline\end{array} \qquad \mathbf{Z}^{-1} = \begin{array}{|c|c|c|c|}\hline 1/a & & & \\ \hline & 1/b & & \\ \hline & & 1/c & \\ \hline & & & 1/d \\ \hline\end{array} \qquad 1.24$$

5. The product

$$\mathbf{Z}\cdot\mathbf{Z}^{-1} = \mathbf{I} \quad \text{and} \quad \mathbf{Z}^{-1}\cdot\mathbf{Z} = \mathbf{I} \qquad 1.25$$

Hence whether the inverse of a matrix has been correctly calculated can be easily checked by multiplying the inverse by the original in any order. The product must be the unit matrix.

6. If the determinant of a square matrix is zero, then its inverse $\mathbf{Z}^{-1}$ does not exist. *2-Matrices whose inverses do not exist* (being rectangular or having zero determinant) *are called "singular" matrices.*

ORDER OF MATRICES

1. In general, the order of *n*-matrices cannot be disturbed. For instance

$$\mathbf{A}\cdot\mathbf{B} \neq \mathbf{B}\cdot\mathbf{A} \quad \text{or} \quad (\mathbf{A}\cdot\mathbf{i})\cdot\mathbf{B} \neq \mathbf{A}\cdot\mathbf{B}\cdot\mathbf{i} \qquad 1.26$$

Exceptions are:

(*a*) If $\mathbf{e}$ and $\mathbf{i}$ are 1-matrices, then

$$\mathbf{e}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{e} \qquad 1.27$$

(*b*) If $\mathbf{A}$ is a 2-matrix and $\mathbf{i}$ is a 1-matrix, then

$$\mathbf{A}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{A}_t \qquad 1.28$$

2. When three 2-matrices are to be multiplied together, as $\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}$, then the multiplication may be performed in any succession as:

(*a*) First $\mathbf{A}\cdot\mathbf{B}$, then $(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C}$.

(*b*) First $\mathbf{B}\cdot\mathbf{C}$, then $\mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C})$.

But the order cannot be interchanged. That is, under (*b*), $\mathbf{B}\cdot\mathbf{C}$ should not be multiplied by $\mathbf{A}$ in the wrong order $(\mathbf{B}\cdot\mathbf{C})\cdot\mathbf{A}$.

3. If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are 2-matrices, then

$$\begin{aligned}(\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C})_t &= \mathbf{C}_t\cdot\mathbf{B}_t\cdot\mathbf{A}_t \\ (\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C})^{-1} &= \mathbf{C}^{-1}\cdot\mathbf{B}^{-1}\cdot\mathbf{A}^{-1}\end{aligned} \qquad 1.29$$

Note that the order is reversed.

MANIPULATION OF MATRIC EQUATIONS

An equation in which each symbol is an n-matrix is called a "matric equation." Each term in a matric equation has the same dimensions. For instance,

1. Each term is a 0-matrix (a "form"):

   (*a*) Equation of power, $P = \mathbf{e}\cdot\mathbf{i}$.
   (*b*) Equation of energy, $T = \frac{1}{2}\mathbf{i}\cdot\mathbf{L}\cdot\mathbf{i}$.

2. Each term is a 1-matrix.

   (*a*) Equation of voltage, $\mathbf{e} = \mathbf{R}\cdot\mathbf{i} + \mathbf{L}\cdot p\mathbf{i} + (\mathbf{S}/p)\cdot\mathbf{i}$.
   (*b*) Equation of current, $\mathbf{i} = \mathbf{Y}\cdot\mathbf{e}$.

3. Each term is a 2-matrix.

   (*a*) Short-circuit impedance, $\mathbf{Z}_1' = \mathbf{Z}_1 - \mathbf{Z}_2\cdot\mathbf{Z}_4^{-1}\cdot\mathbf{Z}_3$.
   (*b*) Law of transformation, $\mathbf{Z}' = \mathbf{C}_t\cdot\mathbf{Z}\cdot\mathbf{C}$.

In a matric equation, only a 2-matrix (or products of n-matrices forming a 2-matrix) can be transferred to the other side of the equation by multiplying both sides by the inverse matrix. For example, let

$$\mathbf{e} = \mathbf{Z}_1\cdot\mathbf{i}_1 + \mathbf{Z}_2\cdot\mathbf{i}_2 \qquad 1.30$$

To solve for $\mathbf{i}_2$, multiply each term by $\mathbf{Z}_2^{-1}$. (It is assumed that the inverse of $\mathbf{Z}_2$ exists.)

$$\mathbf{Z}_2^{-1}\cdot\mathbf{e} = \mathbf{Z}_2^{-1}\cdot\mathbf{Z}_1\cdot\mathbf{i}_1 + \mathbf{Z}_2^{-1}\cdot\mathbf{Z}_2\cdot\mathbf{i}_2$$

But
$$\mathbf{Z}_2^{-1}\cdot\mathbf{Z}_2 = \mathbf{I} \quad \text{and} \quad \mathbf{I}\cdot\mathbf{i}_2 = \mathbf{i}_2$$

Hence
$$\mathbf{Z}_2^{-1}\cdot\mathbf{e} = \mathbf{Z}_2^{-1}\cdot\mathbf{Z}_1\cdot\mathbf{i}_1 + \mathbf{i}_2$$

or
$$\mathbf{i}_2 = \mathbf{Z}_2^{-1}\cdot[\mathbf{e} - \mathbf{Z}_1\cdot\mathbf{i}_1] \qquad 1.31$$

Otherwise matric equations are manipulated in exactly the same manner as ordinary equations. Only the order of symbols should not be disturbed.

EXERCISES

If

$$\mathbf{e} = \begin{array}{|c|c|c|c|}\hline 2 & 3 & -4 & 5 \\ \hline\end{array}$$

$$\mathbf{i} = \begin{array}{|c|c|c|c|}\hline -1 & 5 & 0 & 3 \\ \hline\end{array}$$

$$\mathbf{A} = \begin{array}{|c|c|c|c|}\hline 2 & 5 & 0 & -2 \\ \hline 3 & 8 & 9 & -5 \\ \hline 1 & 5 & 3 & 2 \\ \hline 0 & 1 & 2 & 5 \\ \hline\end{array}$$

$$\mathbf{B} = \begin{bmatrix} r+jx & 3 & x & \left(\frac{d}{dt}\right)^2 \\ 2 & 0 & -x & y^2 \\ x+y & 2 & \frac{d}{dt} & -1 \\ -xy & 3 & 4+2j & 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Find:

1. $\mathbf{e} + \mathbf{i}$.
2. $\mathbf{e} \cdot \mathbf{i}$.
3. $\mathbf{e} \cdot \mathbf{A}$.
4. $\mathbf{e} \cdot \mathbf{A} \cdot \mathbf{i}$.
5. $\mathbf{A}_t$.
6. $\mathbf{A} \cdot \mathbf{B}_t$.
7. $\mathbf{C}^{-1}$.
8. $\mathbf{C}^{-1} \cdot \mathbf{C}$.

9. Given two matric equations with two unknowns $\mathbf{i}_1$ and $\mathbf{i}_2$

$$\mathbf{e}_1 = \mathbf{Z}_1 \cdot \mathbf{i}_1 + \mathbf{Z}_2 \cdot \mathbf{i}_2$$
$$\mathbf{e}_2 = \mathbf{Z}_3 \cdot \mathbf{i}_1 + \mathbf{Z}_4 \cdot \mathbf{i}_2$$

Solve them for $\mathbf{i}_1$.

10. Given three matric equations with three unknowns $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$.

$$\mathbf{e}_1 = \mathbf{Z}_1 \cdot \mathbf{i}_1 + \mathbf{Z}_2 \cdot \mathbf{i}_2 + \mathbf{Z}_3 \cdot \mathbf{i}_3$$
$$\mathbf{e}_2 = \mathbf{Z}_4 \cdot \mathbf{i}_1 + \mathbf{Z}_5 \cdot \mathbf{i}_2 + \mathbf{Z}_6 \cdot \mathbf{i}_3$$
$$\mathbf{e}_3 = \mathbf{Z}_7 \cdot \mathbf{i}_1 + \mathbf{Z}_8 \cdot \mathbf{i}_2 + \mathbf{Z}_9 \cdot \mathbf{i}_3$$

Eliminate $\mathbf{i}_3$ from the third equation so as to leave only two equations with the two unknowns $\mathbf{i}_1$ and $\mathbf{i}_2$.

CHAPTER 2

COMPOUND n-MATRICES*

PARTITION OF MATRICES

(*a*) Any n-matrix can be subdivided into several smaller parts, each part forming a similar n-matrix.

$$\mathrm{i} = \begin{array}{|c|c|c|c|c|}\hline 4 & 5 & 6 & 7 & 8 \\ \hline\end{array} = \begin{array}{|c|c||c|c|c|}\hline 4 & 5 & 6 & 7 & 8 \\ \hline\end{array} = \begin{array}{|c|c|}\hline \mathrm{i}_1 & \mathrm{i}_2 \\ \hline\end{array}$$

$$\mathrm{Z} = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 5 & 6 & \\ \hline 3 & 4 & & & 7 \\ \hline 8 & & 2 & & 3 \\ \hline & 9 & & -5 & \\ \hline -1 & & 2 & & 1 \\ \hline\end{array} = \begin{array}{|c|c||c|c|c|}\hline 1 & 2 & 5 & 6 & \\ \hline 3 & 4 & & & 7 \\ \hline\hline 8 & & 2 & & 3 \\ \hline & 9 & & -5 & \\ \hline -1 & & 2 & & 1 \\ \hline\end{array} = \begin{array}{|c|c|}\hline \mathrm{Z}_1 & \mathrm{Z}_2 \\ \hline \mathrm{Z}_3 & \mathrm{Z}_4 \\ \hline\end{array} \qquad 2.1$$

Matrices in which each element itself is a matrix are called "compound matrices." They are added, multiplied, etc., as ordinary matrices with certain precautions.

(*b*) In taking the transpose of a compound 2-matrix, the transpose of each element is also taken.

$$\mathrm{A} = \begin{array}{|c|c|}\hline \mathrm{A} & \mathrm{B} \\ \hline \mathrm{C} & \mathrm{D} \\ \hline\end{array} \qquad \mathrm{A}_t = \begin{array}{|c|c|}\hline \mathrm{A}_t & \mathrm{C}_t \\ \hline \mathrm{B}_t & \mathrm{D}_t \\ \hline\end{array} \qquad 2.2$$

(*c*) When two n-matrices with a large number of rows and columns are to be multiplied, they are first divided into compound matrices and only the latter are multiplied together. Afterward the component matrices are multiplied as indicated. E.g., in the above example

$$\mathrm{Z}\cdot\mathrm{i} = \overrightarrow{\begin{array}{|c|c|}\hline \mathrm{Z}_1 & \mathrm{Z}_2 \\ \hline \mathrm{Z}_3 & \mathrm{Z}_4 \\ \hline\end{array}} \cdot \begin{array}{|c|}\hline \mathrm{i}_1 \\ \hline \mathrm{i}_2 \\ \hline\end{array}\Big\downarrow = \begin{array}{|c|}\hline \mathrm{Z}_1\cdot\mathrm{i}_1 + \mathrm{Z}_2\cdot\mathrm{i}_2 \\ \hline \mathrm{Z}_3\cdot\mathrm{i}_1 + \mathrm{Z}_4\cdot\mathrm{i}_2 \\ \hline\end{array} = \begin{array}{|c|}\hline \mathrm{e}_1 \\ \hline \mathrm{e}_2 \\ \hline\end{array} \qquad 2.3$$

* *T.A.N.*, Chapters IX and X.

where

$$\mathbf{e}_1 = \mathbf{Z}_1\cdot\mathbf{i}_1 + \mathbf{Z}_2\cdot\mathbf{i}_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 5 & 6 & \\ & & 7 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 14 \\ 32 \end{bmatrix} + \begin{bmatrix} 72 \\ 56 \end{bmatrix} = \begin{bmatrix} 86 \\ 88 \end{bmatrix}$$

$$\mathbf{e}_2 = \mathbf{Z}_3\cdot\mathbf{i}_1 + \mathbf{Z}_4\cdot\mathbf{i}_2 = \begin{bmatrix} 8 & \\ & 9 \\ -1 & \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 & & 3 \\ & -5 & \\ 2 & & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 32 \\ 45 \\ -4 \end{bmatrix} + \begin{bmatrix} 36 \\ -35 \\ 20 \end{bmatrix} = \begin{bmatrix} 68 \\ 10 \\ 16 \end{bmatrix}$$

Hence

$$\mathbf{Z}\cdot\mathbf{i} = \mathbf{e} = \begin{bmatrix} 86 & 88 & 68 & 10 & 16 \end{bmatrix}$$

ELIMINATION OF PERMANENTLY SHORT-CIRCUITED MESHES*

(*a*) In engineering work rarely are as many equations written down as there are variables in the problem. For instance, in electrical network or machinery problems, the equations of permanently short-circuited meshes are left out and only the active mesh equations are handled. The question now arises, how to eliminate superfluous variables from a set of linear equations. (Or, speaking physically, how to eliminate certain meshes. These meshes may or may not contain impressed voltages.)

Mathematically, the problem may also be formulated: How may a large number of variables be eliminated at one step from a set of linear equations instead of one variable being eliminated at a time?

(*b*) Let the n linear equation, say $\mathbf{e} = \mathbf{Z}\cdot\mathbf{i}$, be divided into *two* sets of equations in any arbitrary manner.

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 \\ \mathbf{Z}_3 & \mathbf{Z}_4 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \end{bmatrix}$$

$$\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2$$

$$\mathbf{i} = \mathbf{i}_1 + \mathbf{i}_2 \qquad 2.4$$

$$\mathbf{Z} = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_4$$

* *A.T.E.M.*, p. 76.

where $\mathbf{e}_2$ indicates the impressed voltages and $\mathbf{i}_2$ the currents in the meshes to be eliminated.

As shown in the previous example, the *single* equation $\mathbf{e} = Z\cdot\mathbf{i}$ may then be replaced by *two* equations

$$\mathbf{e}_1 = \mathbf{Z}_1\cdot\mathbf{i}_1 + \mathbf{Z}_2\cdot\mathbf{i}_2 \tag{2.5}$$

$$\mathbf{e}_2 = \mathbf{Z}_3\cdot\mathbf{i}_1 + \mathbf{Z}_4\cdot\mathbf{i}_2 \tag{2.6}$$

(*c*) Let $\mathbf{i}_2$ be eliminated from the second equation. That is, let a set of variables be eliminated. The procedure is exactly the same as if two scalar equations were solved for the two unknowns.

$$\mathbf{Z}_4\cdot\mathbf{i}_2 = \mathbf{e}_2 - \mathbf{Z}_3\cdot\mathbf{i}_1$$

$$\mathbf{i}_2 = \mathbf{Z}_4^{-1}\cdot(\mathbf{e}_2 - \mathbf{Z}_3\cdot\mathbf{i}_1) \tag{2.7}$$

Substituting $\mathbf{i}_2$ into the first equation

$$\mathbf{e}_1 = \mathbf{Z}_1\cdot\mathbf{i}_1 + \mathbf{Z}_2\cdot\mathbf{Z}_4^{-1}\cdot(\mathbf{e}_2 - \mathbf{Z}_3\cdot\mathbf{i}_1)$$

$$\mathbf{e}_1 = \mathbf{Z}_1\cdot\mathbf{i}_1 + \mathbf{Z}_2\cdot\mathbf{Z}_4^{-1}\cdot\mathbf{e}_2 - \mathbf{Z}_2\cdot\mathbf{Z}_4^{-1}\cdot\mathbf{Z}_3\cdot\mathbf{i}_1$$

$$\mathbf{e}_1 = \mathbf{Z}_2\cdot\mathbf{Z}_4^{-1}\cdot\mathbf{e}_2 + (\mathbf{Z}_1 - \mathbf{Z}_2\cdot\mathbf{Z}_4^{-1}\cdot\mathbf{Z}_3)\cdot\mathbf{i}_1$$

Therefore

$$\mathbf{e}_1 - \mathbf{Z}_2\cdot\mathbf{Z}_4^{-1}\cdot\mathbf{e}_2 = (\mathbf{Z}_1 - \mathbf{Z}_2\cdot\mathbf{Z}_4^{-1}\cdot\mathbf{Z}_3)\cdot\mathbf{i}_1 \tag{2.8}$$

This is an equation of the form $\mathbf{e}' = \mathbf{Z}'\cdot\mathbf{i}_1$ but it contains less rows than the original set. The new $\mathbf{Z}'$ of the reduced network is

$$\boxed{\mathbf{Z}' = \mathbf{Z}_1 - \mathbf{Z}_2\cdot\mathbf{Z}_4^{-1}\cdot\mathbf{Z}_3} \tag{2.9}$$

(this may be called the "short-circuit" matrix) and its new impressed voltage is

$$\boxed{\mathbf{e}' = (\mathbf{e}_1 - \mathbf{Z}_2\cdot\mathbf{Z}_4^{-1}\cdot\mathbf{e}_2)} \tag{2.10}$$

(*d*) Solving for $\mathbf{i}_1$

$$\mathbf{i}_1 = \mathbf{Z}'^{-1}\cdot\mathbf{e}' \tag{2.11}$$

That is, the set of currents $\mathbf{i}_1$ is calculated by finding first the inverse of only one matrix $\mathbf{Z}_4$ having less rows and columns than the original matrix, then after several multiplications the inverse of another matrix is found having as many rows and columns as $\mathbf{Z}_1$.

If $\mathbf{i}_1$ is already known, the value of the eliminated currents $\mathbf{i}_2$ is found from equation 2.7:

$$\mathbf{i}_2 = \mathbf{Z}_4^{-1}\cdot(\mathbf{e}_2 - \mathbf{Z}_3\cdot\mathbf{i}_1) \qquad 2.12$$

(*e*) In many cases $\mathbf{e}_2 = 0$ (that is, the eliminated meshes contain no impressed voltages). Then equations 2.11 and 2.12 are simplified to

$$\mathbf{i}_1 = \mathbf{Z}'^{-1}\cdot\mathbf{e}_1 \qquad 2.13$$

$$\mathbf{i}_2 = -\mathbf{Z}_4^{-1}\cdot\mathbf{Z}_3\cdot\mathbf{i}_1 \qquad 2.14$$

(*f*) If the new set of equations $\mathbf{e}' = \mathbf{Z}'\cdot\mathbf{i}_1$ contains several variables, it can again be subdivided into two sets of equations and the above process repeated.

The calculation of inverse matrices is entirely avoided if *one variable at a time* is eliminated. This step is equivalent to the usual star-mesh transformation *that eliminates the meshes one at a time.**

SOLVING A SET OF LINEAR EQUATIONS IN TWO STEPS

(*a*) Given five equations with five unknowns.

$$\begin{aligned}
10 &= i_a + 2i_b - 3i_c + 4i_d + 5i_f \\
9 &= 2i_a + 4i_b + 3i_c + 5i_d - i_f \\
8 &= 3i_a + 4i_b + 5i_c + 2i_d + 3i_f \\
7 &= i_a + 2i_b - 4i_c - 3i_d + 5i_f \\
6 &= 5i_a + i_b - 3i_c + 3i_d + 2i_f
\end{aligned} \qquad 2.15$$

If the five equations are written as $\mathbf{e} = \mathbf{Z}\cdot\mathbf{i}$, then

$$\mathbf{e} = \begin{bmatrix} 10 & 9 & 8 & 7 & 6 \end{bmatrix} \qquad \mathbf{i} = \begin{bmatrix} i_a & i_b & i_c & i_d & i_f \end{bmatrix} \qquad \mathbf{Z} = \begin{bmatrix} 1 & 2 & -3 & 4 & 5 \\ 2 & 4 & 3 & 5 & -1 \\ 3 & 4 & 5 & 2 & 3 \\ 1 & 2 & -4 & -3 & 5 \\ 5 & 1 & -3 & 3 & 2 \end{bmatrix} \qquad 2.16$$

* *T.A.N.*, p. 261.

The problem is to solve for the five unknowns **i**. Three of the unknowns i_c, i_d, and i_f can be eliminated in one step by separating the first two from the last three components so that

$$\mathbf{e}_1 = \begin{array}{|c|c|}\hline 10 & 9 \\ \hline\end{array} \qquad \mathbf{e}_2 = \begin{array}{|c|c|c|}\hline 8 & 7 & 6 \\ \hline\end{array}$$

$$\mathbf{i}_1 = \begin{array}{|c|c|}\hline i_a & i_b \\ \hline\end{array} \qquad \mathbf{i}_2 = \begin{array}{|c|c|c|}\hline i_c & i_d & i_f \\ \hline\end{array} \tag{2.17}$$

$$\mathbf{Z}_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 4 \\ \hline\end{array} \qquad \mathbf{Z}_2 = \begin{array}{|c|c|c|}\hline -3 & 4 & 5 \\ \hline 3 & 5 & -1 \\ \hline\end{array}$$

$$\mathbf{Z}_3 = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline 5 & 1 \\ \hline\end{array} \qquad \mathbf{Z}_4 = \begin{array}{|c|c|c|}\hline 5 & 2 & 3 \\ \hline -4 & -3 & 5 \\ \hline -3 & 3 & 2 \\ \hline\end{array} \tag{2.18}$$

(*b*) If the last three rows and columns are eliminated, the remaining matrix (having two rows and columns) is

$$\boxed{\mathbf{Z}' = \mathbf{Z}_1 - \mathbf{Z}_2 \cdot \mathbf{Z}_4^{-1} \cdot \mathbf{Z}_3} \tag{2.19}$$

$$\mathbf{Z}_4^{-1} = \frac{1}{-182} \times \begin{array}{|c|c|c|}\hline -21 & 5 & 19 \\ \hline -7 & 19 & -37 \\ \hline -21 & -21 & -7 \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline \cancel{0.005} & -0.0274 & -0.104 \\ \hline 0.0384 & -0.104 & 0.203 \\ \hline 0.121 & 0.121 & 0.0384 \\ \hline\end{array} \tag{2.20}$$

$$\mathbf{Z}_2 \cdot \mathbf{Z}_4^{-1} = \begin{array}{|c|c|c|}\hline -0.415 & 0.272 & 1.316 \\ \hline 0.416 & -0.723 & 0.665 \\ \hline\end{array}$$

$$(\mathbf{Z}_2 \cdot \mathbf{Z}_4^{-1}) \cdot \mathbf{Z}_3 = \begin{array}{|c|c|}\hline 5.6 & 0.2 \\ \hline 3.847 & 0.885 \\ \hline\end{array}$$

$$Z' = Z_1 - Z_2 \cdot Z_4^{-1} \cdot Z_3 = \begin{array}{|c|c|}\hline -4.6 & 1.8 \\ \hline -1.847 & 3.115 \\ \hline\end{array} \qquad 2.2$$

(*c*) The new applied voltages are

$$\boxed{e' = e_1 - Z_2 \cdot Z_4^{-1} \cdot e_2} \qquad 2.2$$

Since $Z_2 \cdot Z_4^{-1}$ was already calculated

$$Z_2 \cdot Z_4^{-1} \cdot e_2 = \begin{array}{|c|c|}\hline -6.48 & 2.29 \\ \hline\end{array} \qquad 2.2$$

$$e' = e_1 - Z_2 \cdot Z_4^{-1} \cdot e_2 = \begin{array}{c}\begin{array}{cc} a & b \end{array} \\ \begin{array}{|c|c|}\hline 16.48 & 6.71 \\ \hline\end{array}\end{array} \qquad 2.24$$

(*d*) Hence, *the remaining two equations with two unknowns are*

$$\boxed{e' = Z' \cdot i_1} \qquad 2.25$$

where

$$e' = \begin{array}{|c|c|}\hline 16.48 & 6.71 \\ \hline\end{array} \quad Z' = \begin{array}{|c|c|}\hline -4.6 & 1.8 \\ \hline -1.847 & 3.115 \\ \hline\end{array} \quad i_1 = \begin{array}{|c|c|}\hline i_a & i_b \\ \hline\end{array} \qquad 2.26$$

which can be solved for i_1 as $i_1 = Z'^{-1} \cdot e'$.

The eliminated currents are found by $i_2 = Z_4^{-1}(e - Z_3 \cdot i_1)$.

EXERCISES

1. If

$$A = \begin{array}{|c|c|c|}\hline 2 & 3 & 0 \\ \hline -1 & 4 & -2 \\ \hline 0 & -3 & 1 \\ \hline\end{array} \qquad B = \begin{array}{|c|c|c|}\hline -1 & 2 & 1 \\ \hline -3 & 4 & 2 \\ \hline 2 & 0 & 7 \\ \hline\end{array} \qquad 2.27$$

Find the products $Z_1 \cdot Z_2$; $Z_2 \cdot Z_1$; $Z_{1t} \cdot Z_2$; $Z_1 \cdot Z_{2t}$, where

$$Z_1 = \begin{array}{|c|c|c|}\hline A & & B \\ \hline & B_t & \\ \hline B & & A_t \\ \hline\end{array} \qquad Z_2 = \begin{array}{|c|c|}\hline A_t & \\ \hline & B \\ \hline B & A \\ \hline\end{array} \qquad 2.28$$

2. Find $A_t \cdot A$ and $A \cdot A_t$ with the aid of compound matrices if

$$A = \left[\begin{array}{cccc|ccc|cccc|ccc}
b & & & -b & 1 & & & & & & & a & & \\
 & b & & & & 1 & & & & & & & a & \\
 & & b & & & & 1 & & & & & & & a \\
\hline
1 & & & & & & & 1 & & & -1 & -b & & \\
 & 1 & & & & & & & 1 & & & & -b & \\
 & & 1 & & & & & & & 1 & & & & -b \\
-1 & & & 1 & & & & & & & 1 & b & & \\
\hline
 & & & & -1 & & & & & & & -a & & \\
 & & & & & -1 & & & & & & & -a & \\
 & & & & & & -1 & & & & & & & -a
\end{array}\right] \qquad 2.29$$

3. Solve the equation $e = Z \cdot i$ for i^1 by eliminating *one variable at a time.*

$$e = \begin{bmatrix} e & & & \end{bmatrix} \qquad i = \begin{bmatrix} i^1 & i^2 & i^3 & i^4 \end{bmatrix} \qquad Z = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & p & q \end{bmatrix} \qquad 2.30$$

4. Given five equations with five unknowns, eliminate i^c, i^d, and i^f in one step.

$$\begin{aligned}
2 &= 3i^a + 4i^b - 2i^c + 6i^d - i^f \\
0 &= \qquad 2i^b \qquad + 7i^d \\
-3 &= i^a - i^b + i^c \qquad + i^f \\
0 &= i^a \qquad + i^c + 3i^d \\
1 &= \qquad 3i^b - i^c \qquad + 2i^f
\end{aligned} \qquad 2.31$$

CHAPTER 3

TRANSFORMATION THEORY*

The Primitive Network

(*a*) Let the network of Fig. 3.1 be given having four coils and three meshes. (In going around a mesh, if two coils are connected 1–2, 1–2, then their fluxes are in the same direction; when connected as 1–2, 2–1, their fluxes oppose each other.)

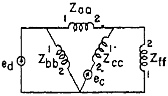


FIG. 3.1. Given network.

The impedance of each coil may be expressed as $R + jX$ or as $R + Lp + 1/pC$ or in other forms. The coils may be parts of a rotating machine or of a vacuum tube, etc. Here they are all called "coils."

The problem is to establish their equations of performance $\mathbf{e} = \mathbf{Z} \cdot \mathbf{i}$ consisting of three linear equations.

(*b*) The method of tensor analysis states:

1. Don't try to set up immediately the three matrices **e**, **Z**, and **i** of this network.

2. First set up **e**, **Z**, and **i** of another network whose analysis is much simpler.

This other network is found by removing all interconnections between the coils and short-circuiting each, as shown in Fig. 3.2. This is called the "primitive network."

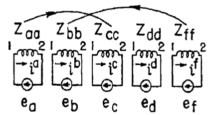


FIG. 3.2. The primitive network of Fig. 3.1.

An impressed voltage e_d in a branch with zero impedance is also assumed to have an impedance Z_{dd} whose value, however, is zero. Similarly, a coil with impedance Z_{bb} is also assumed to have an impressed voltage e_b in series with it, whose value, however, is zero. (The arrows show which coils have mutual impedances and in which direction. It should be noted that Z_{bf} and Z_{ca} are zero.)

* *T.A.N.*, Chapter IV; *A.T.E.M.*, Part II; *G.E.R.*, May, 1935.

(*c*) For the primitive network **e**, **Z**, and **i** are established by simple inspection, as

$$
\mathbf{e} = \begin{array}{c} \begin{array}{ccccc} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} & \mathrm{f} \end{array} \\ \left[\begin{array}{ccccc} & & e_c & e_d & \end{array}\right] \end{array}
\qquad
\mathbf{Z} = \begin{array}{c|ccccc} & \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} & \mathrm{f} \\ \hline \mathrm{a} & Z_{aa} & & Z_{ac} & & \\ \mathrm{b} & & Z_{bb} & & & \\ \mathrm{c} & & & Z_{cc} & & \\ \mathrm{d} & & & & & \\ \mathrm{f} & & Z_{fb} & & & Z_{ff} \end{array}
\qquad 3.1
$$

$$
\mathbf{i} = \begin{array}{c} \begin{array}{ccccc} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} & \mathrm{f} \end{array} \\ \left[\begin{array}{ccccc} i^a & i^b & i^c & i^d & i^f \end{array}\right] \end{array}
$$

All zeros are left out.

In order not to cause confusion, each row and column has an index alongside referring to a particular coil. This index may also be considered a unit vector.

The equation of performance of the primitive network is $\mathbf{e} = \mathbf{Z}\cdot\mathbf{i}$, representing as many linear equations as there are coils. If needed, the equations may be solved for the currents as $\mathbf{i} = \mathbf{Z}^{-1}\cdot\mathbf{e}$.

The Transformation Matrix

The next problem is to establish some relation between the given network and the primitive network.

1. Assume as many currents in the given networks as there are

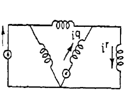


FIG. 3.3. New variables.

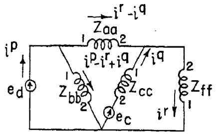


FIG. 3.4. New currents in each coil.

meshes, Fig. 3.3. These currents may be assumed anywhere as long as they are independent of each other.

2. Express the currents flowing in every coil in terms of these new currents with the aid of Kirchhoff's laws (Fig. 3.4).

3. If the primitive network, Fig. 3.2, is compared with this last network, Fig. 3.4, each coil now has two different expressions for the currents flowing in it.

Equate the old expressions with the new expressions for each coil separately by inspecting the two diagrams.

$$\begin{array}{llcll} \text{In coil } Z_{aa} & i^a = & i^r - i^q & = & -i^q + i^r \\ Z_{bb} & i^b = & i^p - i^r + i^q & = & i^p + i^q - i^r \\ Z_{cc} & i_c = & -i^q & = & -i^q \\ Z_{dd} & i^b = & i^p & = & i^p \\ Z_{ff} & i^f = & i^r & = & +i^r \end{array} \qquad 3.2$$

(For each coil the current is written from 1 to 2.)

4. This set of linear equations may be written (analogously to ə = Z·i) as i = C·i′, where the components of C are found by taking :he coefficients of the new currents

$$\mathrm{i} = \begin{array}{ccccc} a & b & c & d & f \\ \hline i^a & i^b & i^c & i^d & i^f \end{array} \qquad \mathrm{C} = \begin{array}{c|ccc} & p & q & r \\ \hline a & & -1 & 1 \\ b & 1 & 1 & -1 \\ c & & -1 & \\ d & 1 & & \\ f & & & 1 \end{array} \qquad 3.3$$

$$\mathrm{i}' = \begin{array}{ccc} p & q & r \\ \hline i^p & i^q & i^r \end{array}$$

This C is called the "connection matrix" since it shows how the coils ıre connected together. Or C represents the relations between the :urrents (the old variables) of the primitive network and the currents the new variables) of the given network.

Ξquations of the Given Network

The equation of the new network e′ = Z′·i′ contains exactly the ame n-matrices in exactly the same order as the equation of the primi- ive network e = Z·i except that they now have different components.

C being known, it is possible to find Z′ and e′ of the given network rom Z and e (i′ is known already) by the following formulas (proof to ollow)

$$\mathrm{Z}' = \mathrm{C}_t\cdot\mathrm{Z}\cdot\mathrm{C} \qquad 3.4$$

$$\mathrm{e}' = \mathrm{C}_t\cdot\mathrm{e} \qquad 3.5$$

Performing the multiplications, the impedance matrix is

$$\mathrm{C}_t\cdot\mathrm{Z} = \begin{bmatrix} & 1 & & 1 & \\ -1 & 1 & -1 & & \\ 1 & -1 & & & 1 \end{bmatrix} \cdot \begin{bmatrix} Z_{aa} & & Z_{ac} & & \\ & Z_{bb} & & & \\ & & Z_{cc} & & \\ & & & & \\ & Z_{fb} & & & Z_{ff} \end{bmatrix} = \begin{bmatrix} & Z_{bb} & & & \\ -Z_{aa} & Z_{bb} & -Z_{ac} - Z_{cc} & & \\ Z_{aa} & -Z_{bb} + Z_{fb} & Z_{ac} & & Z_{ff} \end{bmatrix} \tag{3.6}$$

$$\begin{array}{l} (\mathrm{C}_t\cdot\mathrm{Z})\mathrm{C} = \\ = \mathrm{Z}' = \end{array} \begin{array}{c|c|c|c} & \mathrm{p} & \mathrm{q} & \mathrm{r} \\ \hline \mathrm{p} & Z_{bb} & Z_{bb} & -Z_{bb} \\ \mathrm{q} & Z_{bb} & Z_{aa} + Z_{bb} + Z_{cc} + Z_{ac} & -Z_{bb} - Z_{ac} \\ \mathrm{r} & Z_{fb} - Z_{bb} & Z_{fb} - Z_{bb} - Z_{aa} - Z_{ac} & Z_{ff} + Z_{aa} + Z_{bb} - Z_{fb} \end{array} \tag{3.7}$$

The impressed voltage matrix is

$$\mathrm{e}' = \mathrm{C}_t\cdot\mathrm{e} = \begin{bmatrix} & 1 & & 1 & \\ -1 & 1 & -1 & & \\ 1 & -1 & & & 1 \end{bmatrix} \cdot \begin{bmatrix} \\ \\ e_c \\ e_d \\ \\ \end{bmatrix} = \begin{array}{c} \mathrm{p} \\ \mathrm{q} \\ \mathrm{r} \end{array} \begin{bmatrix} e_d \\ -e_c \\ \\ \end{bmatrix} \tag{3.8}$$

Hence the equations of performance of the network, Fig. 1, are by $\mathrm{e}' = \mathrm{Z}'\cdot\mathrm{i}'$

$$\begin{aligned} e_d &= Z_{bb}i^p + Z_{bb}i^q + (-Z_{bb})i^r \\ -e_c &= Z_{bb}i^p + (Z_{aa} + Z_{bb} + Z_{cc} + Z_{ac})i^q + (-Z_{bb} - Z_{ac})i^r \\ 0 &= (Z_{fb} - Z_{bb})i^p + (Z_{fb} - Z_{bb} - Z_{aa} - Z_{ac})i^q + (Z_{ff} + Z_{aa} + Z_{bb} - Z_{fb})i^r \end{aligned} \tag{3.9}$$

Solutions for Currents and Differences of Potential

Once the equations $\mathrm{e}' = \mathrm{Z}'\cdot\mathrm{i}'$ have been established, they may be subjected to all types of manipulations, usually involving compound matrices. For instance, some of the currents (or their corresponding meshes) may be permanently eliminated by $\mathrm{Z}' = \mathrm{Z}_1 - \mathrm{Z}_2\cdot\mathrm{Z}_4^{-1}\cdot\mathrm{Z}_3$; or the conditions that the various impedances must satisfy in order that some of the currents should remain constant irrespective of how the

loads vary may be investigated. The examples might be continued indefinitely.

The equations can be solved for the currents as

$$\mathbf{i}' = \mathbf{Z}'^{-1} \cdot \mathbf{e}' = \mathbf{Y}' \cdot \mathbf{e}' \qquad 3.10$$

by finding the inverse of $\mathbf{Z}'$ and multiplying it by $\mathbf{e}'$.

Once the currents $\mathbf{i}'$ have been found, then:

1. The differences of potential $\mathbf{e}_c$ existing across each coil are found by

$$\mathbf{e}_c = \mathbf{Z} \cdot \mathbf{C} \cdot \mathbf{i}' \qquad 3.11$$

where $\mathbf{Z} \cdot \mathbf{C}$ already has been calculated in finding $\mathbf{Z}'$ by $\mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C}$.

2. The currents $\mathbf{i}_c$ flowing in each coil are found by

$$\mathbf{i}_c = \mathbf{C} \cdot \mathbf{i}' \qquad 3.12$$

PERMANENCE OF THE METHOD OF REASONING

Of course, in simple problems the ordinary method of analysis is faster than the method shown. The value of the method comes into increasing prominence when:

1. The network becomes more complex.
2. The number of mutual impedances increases.
3. The mutual impedances are asymmetrical.
4. In place of the usual self and mutual impedances, *artificial* types of constants are used, such as "bucking" impedances or "positive-, negative-, and zero-sequence" impedances.
5. In place of the *actual* currents *hypothetical* currents are used such as "symmetrical components" and "load currents."
6. The equation of performance is more complicated than $\mathbf{e} = \mathbf{Z} \cdot \mathbf{i}$.
7. The reference axes are not stationary but move or rotate.
8. The system is not a stationary network but a rotating machine.
9. In place of *circuit* problems, *field* problems are analyzed.
10. The system is not an electrical but a mechanical, optical, elastic, or some other system.

In all such cases the steps shown remain unchanged, and they still give the correct answer automatically, as will be shown subsequently. With ordinary methods of analysis, for each different type of problem a new "trick" has to be invented, each of the tricks requiring sometimes years to learn (and days to forget).

That is, the method of tensors in general does not save time in getting a numerical answer to a particular problem if the problem is attacked

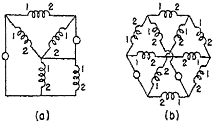
by a specialist. It does save time, however, by avoiding the necessity of inventing a new trick for each new type of problem.

Of course, as the complexity of the system increases, the above steps have to be correspondingly enlarged. *But the given steps still remain the nucleus of the method of attack.*

EXERCISES

1. Find **C** and **Z′** of the networks of Fig. 3.5.

2. Given the bridge network of Fig. 3.6 with mutuals between Z_{aa}-Z_{bb} and Z_{dd}-Z_{ff}. Find the currents and differences of potential in each coil.



(a) (b)

FIG. 3.5.

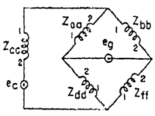


FIG. 3.6.

3. Let **Z** of the primitive network of the rotating machine of Fig. 3.7 be

	a	b	c
a	$r_a + L_a p$	$M_1 p\theta$	$M_2 p + M_3 p\theta$
b	$-M_3 p\theta$	$r_b + L_b p$	$M_4 p + M_5 p\theta$
c	$M_6 p$	$M_7 p$	$r_c + L_c p$

Find **C** and $\mathbf{C}_t\mathbf{Z}\cdot\mathbf{C}$. Write out the two differential equations of the machine.

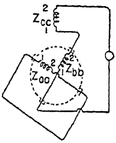


FIG. 3.7.

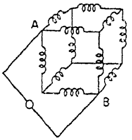


FIG. 3.8.

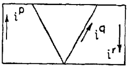
4. Find **C** and **Z′** of the cube of impedances forming Fig. 3.8. What is the impedance between points A and B if each side of the cube is formed by a 1-ohm resistance?

CHAPTER 4

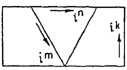
DIFFERENT TYPES OF TRANSFORMATIONS*

Change of Variables

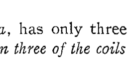
The interconnection of coils is only one of an infinite variety of problems which require the establishment of **C**, that is, which can be treated as a problem in "transformation of the variables **i**." Another such problem occurs where the interconnection of coils remains undisturbed but a new set of currents is introduced. (Other examples will follow.)



(*a*) Old variables.



(*b*) New variables.



(*c*) New currents in each coil.

FIG. 4.1. Change of variables.

Let it be assumed that, in Fig. 3.4 (shown again as Fig. 4.1*a*), the currents i^p, i^q, and i^r are replaced for some reason by another set of three currents i^m, i^n, and i^k as shown in Fig. 4.1*b*. The problem now is to establish the corresponding **C**. With the aid of this **C** then the **e**$'$ and **Z**$'$ of the circuit of Fig. 4.1*a* can be changed to **e**$''$ and **Z**$''$ of Fig. 4.1*c* by the previous formulas.

The steps are exactly the same as before except that now Fig. 4.1*a* forms the "primitive" network instead of Fig. 3.2.

1. Express the currents flowing in each coil in terms of the new currents, as shown in Fig. 4.1*c*.

2. Equate the old expression for current in Fig. 4.1*a* with the new expression in Fig. 4.1*c*. However, since the "primitive" network, Fig. 4.1*a*, has only three variables, *it is now sufficient to equate the expressions in three of the coils only as*

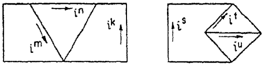
$$\begin{aligned} i^p &= \qquad + i^m + i^n \\ i^q &= - i^k \qquad - i^n \\ i^r &= - i^k \end{aligned} \qquad \mathbf{C} = \begin{array}{c|ccc|} & k & m & n \\ \hline p & & 1 & 1 \\ q & -1 & & -1 \\ r & -1 & & \\ \hline \end{array} \qquad 4.1$$

* *T.A.N.*, Chapters V and VI.

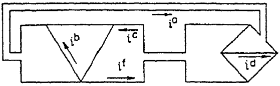
The coefficients of the new currents form the transformation matrix **C**. The remaining work of finding e′′ and Z′′ is purely automatic. If, instead of *actual* branch currents, hypothetical *mesh currents* (flowing in a closed mesh) are assumed, the analysis is the same.

Interconnection of Networks

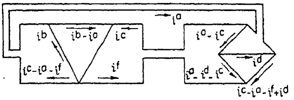
When a complex system consisting of, say, several rotating machines and networks is to be analyzed, it is not necessary to subdivide the whole system into individual coils. It is sufficient to subdivide it into



(*a*) The primitive system.



(*b*) Interconnected system.



(*c*) New currents in each coil.

FIG. 4.2. Interconnection of networks.

several component parts, each consisting of a rotating machine or a network, analyze each separately (if their Z has not yet been found), then interconnect them into the resultant system.

In many cases the Z matrix of all or most component parts has already been calculated on previous occasions, and that work need not be repeated. It is this preservation of previous results for later use in new combinations that is one of the advantages of the tensorial method of attack.

Let, for instance, the two networks of Fig. 4.2*a* be interconnected as shown in Fig. 4.2*b*.

The Primitive System

Before interconnection $\mathbf{Z}$, $\mathbf{e}$, and $\mathbf{i}$ of both systems have been calculated previously as

$$\mathbf{e}' = \begin{array}{c} \begin{array}{ccc} m & n & k \end{array} \\ \left[\begin{array}{ccc} e_m & e_n & e_k \end{array}\right] \end{array} \qquad \mathbf{e}'' = \begin{array}{c} \begin{array}{ccc} s & t & u \end{array} \\ \left[\begin{array}{ccc} e_s & e_t & e_u \end{array}\right] \end{array}$$

$$\mathbf{Z}' = \begin{array}{cc} & \begin{array}{ccc} m & n & k \end{array} \\ \begin{array}{c} m \\ n \\ k \end{array} & \left[\begin{array}{ccc} Z_{mm} & Z_{mn} & Z_{mk} \\ Z_{nm} & Z_{nn} & Z_{nk} \\ Z_{km} & Z_{kn} & Z_{kk} \end{array}\right] \end{array} \qquad \mathbf{Z}'' = \begin{array}{cc} & \begin{array}{ccc} s & t & u \end{array} \\ \begin{array}{c} s \\ t \\ u \end{array} & \left[\begin{array}{ccc} Z_{ss} & Z_{st} & Z_{su} \\ Z_{ts} & Z_{tt} & Z_{tu} \\ Z_{us} & Z_{ut} & Z_{uu} \end{array}\right] \end{array} \tag{4.2}$$

$$\mathbf{i}' = \begin{array}{c} \begin{array}{ccc} m & n & k \end{array} \\ \left[\begin{array}{ccc} i^m & i^n & i^k \end{array}\right] \end{array} \qquad \mathbf{i}'' = \begin{array}{c} \begin{array}{ccc} s & t & u \end{array} \\ \left[\begin{array}{ccc} i^s & i^t & i^u \end{array}\right] \end{array}$$

where $Z_{km} \neq Z_{mk}$.

When the two networks are placed side by side as in Fig. 4.2*a* they form a "primitive system" whose $\mathbf{e}$, $\mathbf{Z}$, and $\mathbf{i}$ matrices (analogously to a primitive system of two coils with no mutuals between them) are

$$\mathbf{e} = \begin{array}{c} \begin{array}{cccccc} m & n & k & s & t & u \end{array} \\ \left[\begin{array}{ccc|ccc} e_m & e_n & e_k & e_s & e_t & e_u \end{array}\right] \end{array} = \left[\begin{array}{c|c} \mathbf{e}' & \mathbf{e}'' \end{array}\right]$$

$$\mathbf{Z} = \begin{array}{cc} & \begin{array}{cccccc} m & n & k & s & t & u \end{array} \\ \begin{array}{c} m \\ n \\ k \\ s \\ t \\ u \end{array} & \left[\begin{array}{ccc|ccc} Z_{mm} & Z_{mn} & Z_{mk} & & & \\ Z_{nm} & Z_{nn} & Z_{nk} & & & \\ Z_{km} & Z_{kn} & Z_{kk} & & & \\ \hline & & & Z_{ss} & Z_{st} & Z_{su} \\ & & & Z_{ts} & Z_{tt} & Z_{tu} \\ & & & Z_{us} & Z_{ut} & Z_{uu} \end{array}\right] \end{array} = \left[\begin{array}{c|c} \mathbf{Z}' & \\ \hline & \mathbf{Z}'' \end{array}\right] \tag{4.3}$$

$$\mathbf{i} = \begin{array}{c} \begin{array}{cccccc} m & n & k & s & t & u \end{array} \\ \left[\begin{array}{ccc|ccc} i^m & i^n & i^k & i^s & i^t & i^u \end{array}\right] \end{array} = \left[\begin{array}{c|c} \mathbf{i}' & \mathbf{i}'' \end{array}\right]$$

(In the right-hand column the use of compound matrices is illustrated. In terms of compound matrices, the analysis of the two networks is analogous to that of two coils in series.)

The Transformation Matrix

(*a*) The **C** matrix is established in exactly the same manner as previously.

1. Assume as many new currents in Fig. 4.2*b* as there are meshes, namely five, i^a, i^b, i^c, i^d, and i^f.

2. Express the currents flowing in each coil in terms of these five currents as shown in Fig. 4.2*c*.

3. Equate the old expressions for currents in Fig. 4.2*a* with the new expressions in Fig. 4.2*c*. Since there are only six old currents in Fig. 4.2*a*, only for their six coils are such equations set up. Hence

$$
\begin{aligned}
i^m &= \quad - i^b \\
i^n &= - i^a + i^b \\
i^k &= \qquad\qquad\qquad + i^f \\
i^s &= \qquad\quad - i^c \\
i^t &= - i^a \quad + i^c \\
i^u &= \qquad\qquad - i^d
\end{aligned}
\qquad
\mathbf{C} = \begin{array}{c} \\ \mathrm{m} \\ \mathrm{n} \\ \mathrm{k} \\ \mathrm{s} \\ \mathrm{t} \\ \mathrm{u} \end{array}
\begin{array}{c}
\begin{array}{ccccc} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} & \mathrm{f} \end{array} \\
\begin{array}{|c|c|c|c|c|}
\hline
 & -1 & & & \\
-1 & 1 & & & \\
 & & & & 1 \\
 & & -1 & & \\
-1 & & 1 & & \\
 & & & -1 & \\
\hline
\end{array}
\end{array}
= \begin{array}{|c|} \hline \mathbf{C}_1 \\ \hline \mathbf{C}_2 \\ \hline \end{array} \qquad 4.4
$$

The coefficients of the new currents form the **C** matrix, and $\mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C}$ gives the impedance matrix of the resultant network, etc.

(*b*) The multiplication may be performed quickly if compound matrices are used. For instance

$$
\mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C} = \begin{array}{|c|c|} \hline \mathbf{C}_{1t} & \mathbf{C}_{2t} \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline \mathbf{Z}' & \\ \hline & \mathbf{Z}'' \\ \hline \end{array} \cdot \begin{array}{|c|} \hline \mathbf{C}_1 \\ \hline \mathbf{C}_2 \\ \hline \end{array} = \mathbf{C}_{1t} \cdot \mathbf{Z}' \cdot \mathbf{C}_1 + \mathbf{C}_{2t} \cdot \mathbf{Z}'' \cdot \mathbf{C}_2
$$

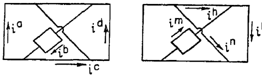
$$
\mathbf{C}_t \cdot \mathbf{e} = \begin{array}{|c|c|} \hline \mathbf{C}_{1t} & \mathbf{C}_{2t} \\ \hline \end{array} \cdot \begin{array}{|c|} \hline \mathbf{e}' \\ \hline \mathbf{e}'' \\ \hline \end{array} = \mathbf{C}_{1t} \cdot \mathbf{e}' + \mathbf{C}_{2t} \cdot \mathbf{e}'' \qquad 4.5
$$

The indicated multiplications and additions are now to be performed.

EXERCISES

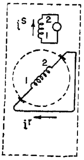
1. Find **C** changing the variables from Fig. 4.3*a* to Fig. 4.3*b*.

2. Find **C** changing the actual branch currents of Fig. 4.4*a* to the hypothetical mesh currents of Fig. 4.4*b*.

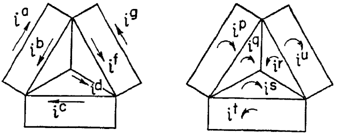


(a) (b)

FIG. 4.3.

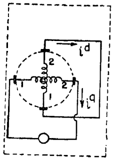


(a) Repulsion motor.



(a) (b)

FIG. 4.4.



(b) Scherbius machine.

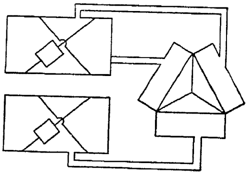
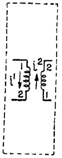


FIG. 4.5.



(c) Transformer.

FIG. 4.6.

3. Find C interconnecting Figs. 4.3a, 4.3b, and 4.4a into Fig. 4.5.

4. Let the transient impedance tensors Z and impressed voltage vectors e of repulsion motor, a Sherbius advancer, and a transformer of Fig. 4.6 be

$$
\mathbf{Z}_1 = \begin{array}{c|c|c|}
 & \text{s} & \text{a} \\
\hline
\text{s} & r_s + L_s p & M \cos \alpha p \\
\hline
\text{a} & M(\cos \alpha p - \sin \alpha p \theta_1) & r_r + L_{rp} \\
\hline
\end{array}
$$

$$
\mathbf{e}_1 = \begin{array}{|c|c|}
\text{s} & \text{a} \\
\hline
e_s 1 & \\
\hline
\end{array}
$$

$$\mathbf{Z}_2 = \begin{array}{c|c|c|} & \mathrm{d} & \mathrm{q} \\ \hline \mathrm{d} & r + Lp & Lp\theta_2 \\ \hline \mathrm{q} & -Lp\theta_2 & r + Lp \\ \hline \end{array} \qquad \mathbf{e}_2 = \begin{array}{|c|c|} \mathrm{d} & \mathrm{q} \\ \hline & e_d 1 \\ \hline \end{array}$$

$$\mathbf{Z}_3 = \begin{array}{c|c|c|} & 1 & 2 \\ \hline 1 & r_1 + L_1 p & M_{12}p \\ \hline 2 & M_{12}p & r_2 + L_2 p \\ \hline \end{array} \qquad \mathbf{e}_3 = \begin{array}{|c|c|} 1 & 2 \\ \hline & \\ \hline \end{array}$$

where $p = d/dt$, 1 = Heaviside unit function, and $p\theta$ = velocity of rotor.

If the three systems are interconnected as shown in Fig. 4.7, what are its transient $\mathbf{Z}'$ and $\mathbf{e}'$?

5. If the impedance tensor of the triade tube of Fig. 4.8 is*

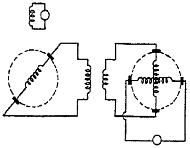


FIG. 4.7.

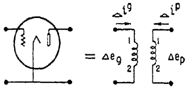


FIG. 4.8.

$$\mathbf{Z} = \frac{1}{1 - \mu_g \mu_p} \begin{array}{c|c|c|} & \mathrm{g} & \mathrm{p} \\ \hline \mathrm{g} & r_g & -\mu_g r_p \\ \hline \mathrm{p} & -\mu_p r_g & r_p \\ \hline \end{array}$$

what is the $\mathbf{Z}'$ of the degenerative feedback amplifier of Fig. 4.9?

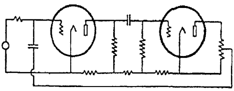


FIG. 4.9.

* *T.A.N.*, Chapter XV.

CHAPTER 5

REACTANCE CALCULATION OF ARMATURE WINDINGS*

Types of Reactances

An a-c. armature winding has several types of reactances such as

1. Total *air-gap* reactance due to *all* the fluxes produced by the winding.

2. *Fundamental* reactance due to *sinusoidal* part of the total flux.

3. *Differential-leakage* reactance due to the difference of the above two fluxes.

4. *Harmonic* reactance due to any of the space *harmonic* fluxes, such as third, fifth, eleventh, etc., harmonics.

With standard methods the calculation of each of the above reactances is time-consuming, and for irregular windings it is prohibitive. To find the fundamental reactance a Fourier analysis of the flux wave is required, for the total air-gap reactance a summation process, a different one for each type of winding, and so on. The tensorial method of attack makes a clean sweep of all these difficulties, no matter how irregular the winding, as long as the air gap is assumed to be uniform.

The Primitive Winding

The steps are exactly the same as for any network:

1. Remove all interconnections between the coils, leaving the "primitive" winding consisting of a large number of isolated coils. Each coil may embrace any number of slots and may have any number of turns. The slots may be unevenly spaced.

2. Calculate the self and mutual reactances of the *individual* coils by the formulas given in Table I.

With standard windings, equidistant coils have the same mutual inductances; hence usually half the reactances repeat themselves. For instance, for a six-coil winding the reactances between winding 1 and the other five coils are shown in Fig. 5.1*a*.

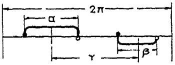
The reactances of winding 2 with the other windings are the same, except that they are shifted by one element, as shown in Fig. 5.1*b*.

* *T.A.N.*, Chapter XII.

TABLE I

MUTUAL-REACTANCE FORMULAS OF TWO ARBITRARY COILS

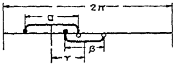
TOTAL AIR-GAP REACTANCE



$\left(2\pi - \frac{\alpha+\beta}{2}\right) \geq \gamma \geq \frac{\alpha+\beta}{2}$

COILS OUTSIDE

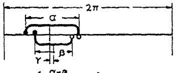
$X_{\alpha\beta} = -k\alpha\beta$ ohms



$\frac{\alpha+\beta}{2} \geq \gamma \geq \frac{\alpha-\beta}{2}$

COILS COUPLED

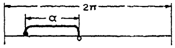
$X_{\alpha\beta} = -k[\alpha\beta - \pi(\alpha+\beta-2\gamma)]$ ohms



$\gamma \leq \frac{\alpha-\beta}{2}$; $\alpha \geq \beta$

COILS INSIDE

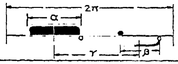
$X'_{\alpha\beta} = -k(\alpha\beta - 2\pi\beta)$ ohms



SELF-REACTANCE

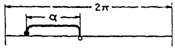
$X_{\alpha\alpha} = -k(\alpha^2 - 2\pi\alpha)$ ohms

SINUSOIDAL REACTANCES



n-TH HARMONIC MUTUAL

$X_{\alpha\beta} = 8K\frac{1}{n^2}\sin\frac{n}{2}\alpha\,\sin\frac{n}{2}\beta\,\cos n\gamma$ ohms



n-TH HARMONIC SELF

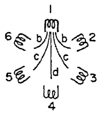
$X_{\alpha\alpha} = 8K\frac{1}{n^2}\left(\sin\frac{n}{2}\alpha\right)^2$ ohms

$K = (2\pi f)0.2N_\alpha N_\beta L\,\frac{R}{\Delta}\,\frac{1}{p}\,10^{-8}$

α, β = Span of coils in electrical radians
γ = Radians between centers of coils

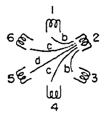
N_α, N_β = No. of turns in coils
p = No. of 2π along armature
f = Frequency of current

R = Radius of armature
Δ = Length of airgap
L = Length of stacking in cm.



	1	2	3	4	5	6
1	a	b	c	d	c	b

FIG. 5.1*a*. Mutuals between winding one and others.



	1	2	3	4	5	6
2	b	a	b	c	d	c

FIG. 5.1*b*. Mutuals between winding two and others.

Hence, the reactance matrix **Z** of the primitive network with six coils is

$$\mathbf{Z} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{c|c|c|c|c|c} a & b & c & d & c & b \\ \hline b & a & b & c & d & c \\ \hline c & b & a & b & c & d \\ \hline d & c & b & a & b & c \\ \hline c & d & c & b & a & b \\ \hline b & c & d & c & b & a \end{array}\right] \end{array} \qquad 5.1$$

The manner of repetition of the first row should be noted.

With a large number of coils, say 72, it is necessary to express **Z** as a compound matrix for easier manipulation. *Even in terms of compound matrices the same pattern repeats.*

$$\mathbf{Z} = \left[\begin{array}{cc|cc|cc} a & b & c & d & c & b \\ b & a & b & c & d & c \\ \hline c & b & a & b & c & d \\ d & c & b & a & b & c \\ \hline c & d & c & b & a & b \\ b & c & d & c & b & a \end{array}\right] = \left[\begin{array}{c|c|c} \mathbf{A} & \mathbf{B} & \mathbf{B}_t \\ \hline \mathbf{B}_t & \mathbf{A} & \mathbf{B} \\ \hline \mathbf{B} & \mathbf{B}_t & \mathbf{A} \end{array}\right] \qquad 5.2$$

Note the appearance of transposed matrices.

(In calculating individual reactances, it is customary to assume the reactance of a single full-pitch coil as unity and express the reactance of all others in terms of that.)

The Transformation Matrix

Since the coils are practically always connected in series only (to form, say, three phases), there are as many new variables as there are windings. The **C** matrix can be established by simple inspection without writing down the set of current equations $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$. For instance, for the capacitor motor winding with sixteen coils (Fig. 5.2) where the pitch of each coil is different, **C** is

The reactance $\mathbf{Z}'$ of the resultant winding is found by the formula

$$\mathbf{Z}' = \mathbf{C}_t\cdot\mathbf{Z}\cdot\mathbf{C} \qquad 5.3$$

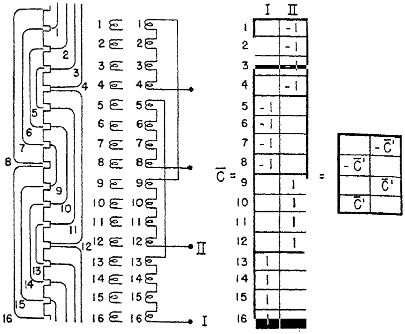


FIG. 5.2. Capacitor motor winding and its connection matrix.

giving the *self* and *mutual* reactances of the various windings. It should be calculated with the aid of compound matrices.

Labor-Saving Devices

Because of the simplicity of the connections, numerous labor-saving devices may be used. For instance:

1. The resultant $\mathbf{Z}'$ should be found in several steps instead of one, namely: (*a*) first interconnect only neighboring coils by $\mathbf{C}_1$; (*b*) then interconnect them into phases by $\mathbf{C}_2$; (*c*) in case of double windings, the phases may be interconnected in various manners. For them establish a separate $\mathbf{C}_3$.

2. The neighboring coils may be interconnected without going through the process of $\mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C}$ as follows.

If $\mathbf{Z}$ is subdivided into compound matrices as suggested by the neighboring coils, then the new $\mathbf{Z}'$ is found by simply adding up the elements

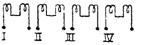


FIG. 5.3. Interconnecting neighboring coils only.

of each compound matrix. For instance, in the case of eight coils, let two neighboring coils be interconnected as shown in Fig. 5.3.

$$
Z = \begin{array}{c|cccc|cccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
1 & a & b & c & d & e & d & c & b \\
2 & b & a & b & c & d & e & d & c \\
\hline
3 & c & b & a & b & c & d & e & d \\
4 & d & c & b & a & b & c & d & e \\
\hline
5 & e & d & c & b & a & b & c & d \\
6 & d & e & d & c & b & a & b & c \\
\hline
7 & c & d & e & d & c & b & a & b \\
8 & b & c & d & e & d & c & b & a \\
\hline
\end{array}
\qquad
Z' = \begin{array}{c|cccc|}
 & \mathrm{I} & \mathrm{II} & \mathrm{III} & \mathrm{IV} \\
\hline
\mathrm{I} & A & B & C & B \\
\mathrm{II} & B & A & B & C \\
\mathrm{III} & C & B & A & B \\
\mathrm{IV} & B & C & B & A \\
\hline
\end{array}
\tag{5.4}
$$

$$
\begin{aligned}
A &= 2a + 2b \\
B &= 2c + b + d \\
C &= 2e + 2d
\end{aligned}
$$

After the neighboring coils only have been interconnected, the new $\mathbf{Z}'$ has half the number of rows as before. The new components are found by adding all the elements within a block. Note that the pattern in $\mathbf{Z}'$ still repeats.

After this step, the coils may be interconnected into phases by a $\mathbf{C}$.

EXERCISES

1. Find $\mathbf{C}$ for the double winding for a turboalternator with 42 coils, Fig. 5.4, and express it as a compound tensor.

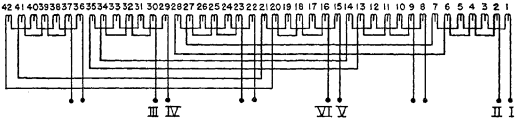


FIG. 5.4.

2. When the resultant $\mathbf{Z}'$ of the above six windings are

$$
Z' = \begin{array}{c|cccccc|}
 & \mathrm{I} & \mathrm{II} & \mathrm{III} & \mathrm{IV} & \mathrm{V} & \mathrm{VI} \\
\hline
\mathrm{I} & a & b & c & d & d & c \\
\mathrm{II} & b & a & b & c & d & d \\
\mathrm{III} & c & b & a & b & c & d \\
\mathrm{IV} & d & c & b & a & b & c \\
\mathrm{V} & d & d & c & b & a & b \\
\mathrm{VI} & c & d & d & c & b & a \\
\hline
\end{array}
$$

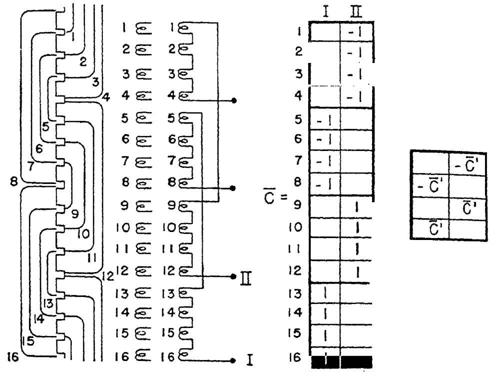


FIG. 5.2. Capacitor motor winding and its connection matrix.

giving the *self* and *mutual* reactances of the various windings. It should be calculated with the aid of compound matrices.

Labor-Saving Devices

Because of the simplicity of the connections, numerous labor-saving devices may be used. For instance:

1. The resultant Z' should be found in several steps instead of one, namely: (*a*) first interconnect only neighboring coils by $\mathbf{C}_1$; (*b*) then interconnect them into phases by $\mathbf{C}_2$; (*c*) in case of double windings, the phases may be interconnected in various manners. For them establish a separate $\mathbf{C}_3$.

2. The neighboring coils may be interconnected without going through the process of $\mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C}$ as follows.

If $\mathbf{Z}$ is subdivided into compound matrices as suggested by the neighboring coils, then the new Z' is found by simply adding up the elements

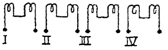


FIG. 5.3. Interconnecting neighboring coils only.

of each compound matrix. For instance, in the case of eight coils, let two neighboring coils be interconnected as shown in Fig. 5.3.

CHAPTER 6

THE LAWS OF TRANSFORMATION*

Definition of a "Tensor"

(*a*) When a network with n meshes is given, instead of saying that the network has n currents, $i^a, i^b \cdots i^n$, and n voltages, $e_a, e_b \cdots e_n$, and n^2 impedances, $Z_{aa}, Z_{ab} \cdots Z_{an}$, it is said that the network has only *one* current $\mathbf{i}$, *one* voltage $\mathbf{e}$, and *one* impedance $\mathbf{Z}$, while the individual currents, voltages, and impedances are simply *elements* of the matrices $\mathbf{i}$, $\mathbf{e}$, and $\mathbf{Z}$.

Suppose that instead of *one* n-mesh network, there are a *large number* of n-mesh networks, each containing the same coils but interconnected in a different manner. With each network there is associated at least one $\mathbf{i}$, $\mathbf{e}$, and $\mathbf{Z}$ matrix (with each network there are actually associated a large number of $\mathbf{i}$, $\mathbf{e}$, and $\mathbf{Z}$ matrices, depending upon where the currents are assumed to flow).

Now instead of saying that there are as many current-matrices $\mathbf{i}'$, $\mathbf{i}''$, $\mathbf{i}''' \cdots$ as there are networks, it is said that there exists only *one physical entity*, a current vector $\mathbf{i}$, whose projections along the various reference frames are the various 1-matrices.

(*b*) This figure of speech is analogous to the statement that the velocity vector $\mathbf{v}$ of a point is the 1-matrix

$$\mathbf{v}' = \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline\end{array}$$

along one reference frame, a different matrix

$$\mathbf{v}'' = \begin{array}{|c|c|c|}\hline 1.5 & 2.5 & 4.5 \\ \hline\end{array}$$

along another frame. Even though the projections vary with the reference frame assumed, the entity $\mathbf{v}$ itself is unchanged.

The key to this definition is the fact that it is possible to find the components of $\mathbf{v}$ (*or* $\mathbf{i}$) *in any reference frame from the components on another frame with the aid of a group of transformation matrices* $\mathbf{C}$ *by a definite formula.*

* *T.A.N.*, Chapter VII.

If the group of **C** is not available, the different n-matrices cannot be changed into each other, are independent from each other, and hence do not form the projections of a single physical entity. A similar statement applies to all **e** and **Z** matrices.

Hence a collection of n-way matrices forms a physical entity, a "tensor of valence n" if with the aid of a group of transformation matrices **C** *they can be changed into one another.*

(*c*) A "tensor of valence 1" like **e** and **i** (represented on each reference frame by a 1-matrix) is called a "vector." A "tensor of valence 0" like power (P) and energy (T) is called a "scalar." Tensors of other valence have no special names. **Z** is then a "tensor of valence 2," the so-called "impedance tensor."

(*d*) A tensor is transformed with the aid of as many **C** (or $\mathbf{C}_t$ or $\mathbf{C}^{-1}$ or $\mathbf{C}_t^{-1}$) as its valence. Hence **e** and **i** require one **C**, **Z** requires two **C**'s, P requires no **C**'s. Because of this "chemical" property of a tensor of attracting a different number of **C**'s, the expression "tensor of valence n" originated. Many writers, though, still call it "tensor of rank n."

It is often said that *a tensor is a matrix with a definite law of transformation.* Actually a tensor is a physical entity, and its projections are the n-way matrices. A tensor differs from a matrix in the same manner as a vector of conventional vector analysis differs from a complex number $2 + 3j$.

Why "Tensors"?

(*a*) The question now arises: Why is it necessary to say that the **e**, **i**, **Z**, etc., matrices of all systems with n coils are only different aspects of the physical entities **e**, **i**, **Z**? What is the advantage of this figure of speech from a practical point of view?

When it is said that the matrices of a particular system are tensors, it automatically follows that *all equations associated with this system are exactly the same in terms of tensors as the equations of a group of physically analogous systems.* If the equation of voltage of one system has been found to be, say, $\mathbf{e} = \mathbf{R}\cdot\mathbf{i} + \mathbf{L}\cdot p\mathbf{i} + (1/p\mathbf{C})\cdot\mathbf{i}$, then if the symbols are tensors, it *automatically* follows that the equations of voltage of every other physically analogous system is exactly the same. If the equation of torque of one system has been found to be $f = \mathbf{i}\cdot\mathbf{G}\cdot\mathbf{i}$, then for every other analogous system the same equation of torque holds true automatically. (Of course, for every system the *components* of the tensors are different.) On the other hand, if the symbols in the equation of $f = \mathbf{i}\cdot\mathbf{G}\cdot\mathbf{i}$ are matrices (that is, if they have not been proved to be tensors), then *this equation is not valid for any other system except for the*

one for which it has been established and every particular system may have an entirely different equation of torque in terms of matrices.

(*b*) What if the equations of a large number of different systems are identical in terms of tensors? Does that fact contribute to simplify the analysis of the large variety of engineering structures?

Yes, it does; and it is just this resulting simplification that underlies the method of reasoning of this treatise. It is advocated here that:

1. Since the equations in terms of tensors are the same for a large number of physically analogous systems, it seems logical that only one of them need be analyzed in detail. Hence select one system whose analysis is simple and establish all the *tensors* of this system (the "primitive" system) and the desired equation of performance in terms of tensors.

2. To find the tensors of any particular system it is then only necessary to find the particular transformation matrix **C** (one aspect of the "transformation tensor" **C**) that differentiates the given system from the primitive system.

3. Once **C** is found, the tensors of the given system can be established by routine laws of transformation.

4. When the components of the tensors of the given system have been found, the sought equation of performance is a copy of that of the primitive system.

(*c*) Of course it is possible to go through the above steps without mentioning the word "tensor," just talking about the "**Z** matrix of the old system" and "**Z** matrix of the new system," the "transformation matrix **C**" and the "law of transformation of **Z**," etc. Nevertheless, *the method of reasoning is that of tensor analysis, whether it is so stated or not. A matrix has no inherent law of transformation; a tensor has such a law.*

Behind the above reasoning looms the all-important question: What is meant by "physically analogous systems" that have the same equations of performance? That is, what systems have a common **C** tensor? This question brings into the foreground the concept of group that was treated in "Tensor Analysis of Networks," Chapter XI.

(*d*) The above-mentioned problem of establishing equations of performance in a simple manner is only one of the numerous examples that show the utility of tensorial concepts. Since mathematical symbols cannot be measured by instruments, only physical entities, the question of what mathematical symbols in the equations do or do not correspond to measurable quantities underlies the foundations of all physical sciences. The word "tensor" is just another expression for "measurable physical entity," and tensor analysis changes the symbols

of a lifeless mathematical equation into living entities. Its concepts and philosophy show, for instance, how to establish stationary equivalent networks that duplicate in some manner the performance of rotating machinery, thereby allowing otherwise difficult measurements to be made conveniently on a stationary network. The general criterion of whether an equation contains only measurable concepts is implied in the basic principle of physics (the so-called first principle of relativity) stating that *all laws of nature are tensor equations*, that is, equations in which each symbol is a tensor.

The Law of Transformation of e

The law of transformation of the voltage vector **e** may be found from the physical fact that in going from one reference frame to another *the instantaneous power input* $\mathbf{e}\cdot\mathbf{i}$ *(a linear form) remains unchanged, or "invariant."* That is

$$P = P' \quad \text{or} \quad \mathbf{e}\cdot\mathbf{i} = \mathbf{e}'\cdot\mathbf{i}' \tag{6.1}$$

This relation is the physical link that connects all networks together.

Now let the currents change from **i** to **i′** by

$$\boxed{\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'} \tag{6.2}$$

Substituting, $\mathbf{e}\cdot\mathbf{C}\cdot\mathbf{i}' = \mathbf{e}'\cdot\mathbf{i}'$

Cancelling **i′** $\mathbf{e}\cdot\mathbf{C} = \mathbf{e}'$

Hence

$$\boxed{\mathbf{e}' = \mathbf{C}_t\cdot\mathbf{e}} \tag{6.3}$$

and

$$\boxed{\mathbf{e} = \mathbf{C}_t^{-1}\cdot\mathbf{e}'} \tag{6.4}$$

It should be noted that, even though both **e** and **i** are vectors, they are transformed to a new reference frame in a different manner. But both being tensors of valence 1, they require **C** once only.

The Law of Transformation of Z

Tensor analysis requires that if the equation of a system in one reference frame has the form $\mathbf{e} = \mathbf{Z}\cdot\mathbf{i}$ it should have the same form in every other frame. This property will give the law of transformation of **Z**. In the old reference frame let

$$\boxed{\mathbf{e} = \mathbf{Z}\cdot\mathbf{i}} \tag{6.5}$$

Express **i** and **e** along the new reference frame. That is, replace **i** by $\mathbf{C}\cdot\mathbf{i}'$ and **e** by $\mathbf{C}_t^{-1}\cdot\mathbf{e}'$.

$$\mathbf{C}_t^{-1}\cdot\mathbf{e}' = \mathbf{Z}\cdot\mathbf{C}\cdot\mathbf{i}'$$

Multiplying both sides by $\mathbf{C}_t$

$$\mathbf{e}' = \mathbf{C}_t\cdot\mathbf{Z}\cdot\mathbf{C}\cdot\mathbf{i}'$$

If the following definition is introduced as the law of transformation of **Z**

$$\boxed{\mathbf{C}_t\cdot\mathbf{Z}\cdot\mathbf{C} = \mathbf{Z}'} \qquad 6.6$$

then the equation in the new reference frame becomes

$$\boxed{\mathbf{e}' = \mathbf{Z}'\cdot\mathbf{i}'} \qquad 6.7$$

The equation of the new system is of the same form as that of the old system, equation 6.5.

The inverse of **Z**, namely $\mathbf{Z}^{-1}$, may be denoted by a separate symbol **Y** so that $\mathbf{i} = \mathbf{Z}^{-1}\cdot\mathbf{e} = \mathbf{Y}\cdot\mathbf{e}$. It is called the "admittance tensor." Its law of transformation is (derived analogously to that of **Z**)

$$\mathbf{Y}' = \mathbf{C}^{-1}\cdot\mathbf{Y}\cdot\mathbf{C}_t^{-1} \qquad 6.8$$

The Transformation Tensor C

The collection of all possible transformation matrices, called the "transformation tensor **C**," is the key to tensor analysis. It is a tensor of valence 2. (Its law of transformation will be derived presently.) **C** *represents the relation between the old and the new reference frames.* Because of that fact **C** differs from **Z** in the respect that, while on both sides of **Z** the same reference axes are written (either both are the old or both the new axes), on the left-hand side of **C** are always written the old axes, on its upper part the new axes.

When coils, beams, wheels, etc., are connected into an engineering structure, the constrained reference axes are ignored; hence in most problems **C** is not square, but rectangular. A study of the missing axes (the "dual" axes) is undertaken in *T.A.N.*, Chapters XIV and XVI.

When **C** is not square (or it is square but its determinant is zero), its inverse $\mathbf{C}^{-1}$ cannot be found. Then **C** is singular and the corresponding transformation is called "singular" transformation. All laws of transformation that do not require $\mathbf{C}^{-1}$ remain valid, however.

The "Group" Property*

If the variables have been changed from $\mathbf{i}$ to $\mathbf{i}'$ by $\mathbf{C}_1$, then from $\mathbf{i}'$ to $\mathbf{i}''$ by $\mathbf{C}_2$, then again from $\mathbf{i}''$ to $\mathbf{i}'''$ by $\mathbf{C}_3$, etc., the successive transformations may be performed in one step with the aid of one transformation tensor

$$\mathbf{C} = \mathbf{C}_1 \cdot \mathbf{C}_2 \cdot \mathbf{C}_3 \cdots \tag{6.9}$$

This important property of $\mathbf{C}$ is called the "group property." Practically all engineering problems consist of two or more successive transformations such as:

1. Interconnect coils into a network by $\mathbf{C}_1$.
2. Neglect magnetizing current by $\mathbf{C}_2$.
3. Introduce symmetrical component by $\mathbf{C}_3$.

The Law of Transformation of C

Let two reference frames be given, and let $\mathbf{C}_2^1$ transform $\mathbf{i}^1$ to $\mathbf{i}^2$ as

$$\mathbf{i}^1 = \mathbf{C}_2^1 \cdot \mathbf{i}^2 \tag{6.10}$$

Now let two other reference frames be introduced, and let $\mathbf{i}^1$ be changed to $\mathbf{i}^3$ by $\mathbf{i}^1 = \mathbf{C}_3^1 \cdot \mathbf{i}^3$ (to that of system 3) and $\mathbf{i}^2$ to $\mathbf{i}^4$ by $\mathbf{i}^2 = \mathbf{C}_4^2 \cdot \mathbf{i}^4$. The question now is how to find $\mathbf{C}_4^3$ changing $\mathbf{i}^3$ to $\mathbf{i}^4$.

Substitute $\mathbf{i}^1$ and $\mathbf{i}^2$ into equation 6.10.

$$\mathbf{C}_3^1 \cdot \mathbf{i}^3 = \mathbf{C}_2^1 \cdot (\mathbf{C}_4^2 \cdot \mathbf{i}^4)$$

Multiplying both sides by the inverse of $\mathbf{C}_1^3$

$$\mathbf{i}^3 = (\mathbf{C}_3^1)^{1-} \cdot \mathbf{C}_2^1 \cdot \mathbf{C}_4^2 \cdot \mathbf{i}^4$$

Writing it as $\mathbf{i}^3 = \mathbf{C}_4^3 \cdot \mathbf{i}^4$ it follows that

$$\boxed{\mathbf{C}_4^3 = (\mathbf{C}_3^1)^{-1} \cdot (\mathbf{C}_2^1 \cdot \mathbf{C}_4^2)} \tag{6.11}$$

Hence, in transforming a $\mathbf{C}$, two other $\mathbf{C}$'s are needed (not one) and the inverse of one has to be known.

The Number of Meshes in a Network†

A network may consist of several independent "subnetworks" (S in number) with no physical connections between them.

* *T.A.N.*, Chapter XI.

† *T.A.N.*, p. 72 and Chapters XIV–XVI.

A network consists of two component parts: (1) coils (C in number); (2) junctions (J in number).

(Two junctions connected, where the two ends of a coil meet, by an impedanceless wire form only one junction.)

The minimum number of closed circuits, or meshes (M in number), is found by the formula

$$M = C - (J - S) \tag{6.12}$$

In Fig. 6.1 there are: (1) two subnetworks; (2) seven coils; (3) four junctions.

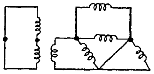


FIG. 6.1.

Hence the number of meshes is

$$M = 7 - (4 - 2) = 5$$

The number $J - S$ (number of junctions minus number of subnetworks) is an important concept called the number of "junction pairs" (P in number). In terms of them

$$\boxed{C = M + P} \tag{6.13}$$

Number of coils = number of meshes + number of junction pairs

CHAPTER 7

EQUATIONS OF CONSTRAINT AS TRANSFORMATIONS

Two Examples of Equations of Constraint

Rarely are the changes from old to new currents stated in a clear-cut manner as $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$. In many cases the distinction between the old and new currents (variables) is hidden and their separation is made by the creation of two physical systems (actually existing or hypothetical) to which the two sets of variables may be attributed.

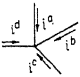


FIG. 7.1.

(*a*) A relation between currents (or the variables) is called an "equation of constraint." For instance, Kirchhoff's first law (Fig. 7.1), "the sum of the currents entering a junction is zero,"

$$i^a + i^b + i^c + i^d = 0 \qquad 7.1$$

represents such an equation since it puts a constraint upon the values that the currents may assume.

If a network has n junctions, the number of such equations (completely representing the interconnections) is $n - 1$.

That is, *the manner of interconnection of a set of C coils into M meshes and J junctions may be represented in two different ways:* (1) with the aid of the C equations of transformations $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$ representing a transformation from an unconstrained (the "primitive") network to the given (constrained) network; (2) or with the aid of P "equations of constraint" $\mathbf{B} \cdot \mathbf{i} = 0$ between the branch currents of the given network, or between the currents of the unconstrained network.

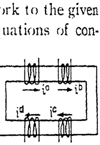


FIG. 7.2.

These two manners of expression are equivalent, and one can be changed into the other, as will be shown presently.

(*b*) Another example, where an equation of constraint $\mathbf{B} \cdot \mathbf{i} = 0$ is set up between the currents of the unconstrained network, is a transformer network (Fig. 7.2) where it is customary to neglect the magnetizing current by assuming that the sum of the m.m.f.'s around the closed magnetic circuit is zero.

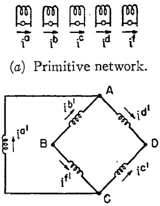
$$n_a i^a + n_b i^b + n_c i^c + n_d i^d = 0 \qquad 7.2$$

(n_a is the number of turns of coil a.) This is also an equation of constraint between the currents of the unconstrained network.

As many such equations may be written as there are closed magnetic circuits in the system. They also can be changed to the alternative form $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$. This change is equivalent to the statement that $\mathbf{i}$ flows in the unconstrained network and $\mathbf{i}'$ flows in a constrained network (which is not yet known).

The problem now is how to express an equation $\mathbf{B}\cdot\mathbf{i} = 0$ or $\mathbf{B}\cdot\mathbf{i}' = 0$ as $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$.

The purpose of establishing a $\mathbf{C}$ *is to make it possible to transform the equation of the unconstrained network, say* $\mathbf{e} = \mathbf{Z}\cdot\mathbf{i}$, *to that of the constrained network* $\mathbf{e}' = \mathbf{Z}'\cdot\mathbf{i}'$ *by the routine laws of tensor analysis.*



(a) Primitive network.

(b) Interconnected network.

FIG. 7.3.

Constraints as "Transformations"

(a) Suppose that a primitive network of five coils is given (Fig. 7.3a) having five *independent* currents, $i^a \cdots i^f$. Each current may assume any value it pleases, and the system is unconstrained.

The effect of interconnecting the same coils into the network of Fig. 7.3b is to prevent the currents in the coils from assuming any value they please. Kirchhoff's first law introduces $4 - 1 = 3$ constraints (where 4 is the number of junctions) between the branch currents, namely:

$$\begin{array}{lr} \text{The constraint of junction } A \text{ is} & i^{a'} + i^{b'} - i^{d'} = 0 \\ \text{" \quad " \quad " \quad " \quad } B \text{ "} & - i^{b'} - i^{f'} = 0 \\ \text{" \quad " \quad " \quad " \quad } C \text{ "} & i^{f'} - i^{a'} - i^{c'} = 0 \end{array} \qquad 7.3$$

In terms of matrices these equations can be written as a matric equation $\mathbf{B}\cdot\mathbf{i}' = 0$, where

$\mathbf{B} =$ (7.4)

	a'	b'	c'	d'	f'
A	1	1		−1	
B		−1			1
C	−1		1		1

It should be expressly noted that in the primitive network unprimed currents flow, while in the interconnected network primed currents flow.

(*b*) The above equations state that the currents in the coils depend upon each other. Let their relations be stated in a slightly different form. Let each equation state that *one of the currents depends upon the others*, by carrying all but one of the currents to the right-hand side of each equation

$$\begin{aligned} i^{d'} &= i^{a'} + i^{b'} \\ i^{f'} &= -\, i^{b'} \\ i^{c'} &= i^{f'} - i^{a'} = -\, i^{b'} - i^{a'} \end{aligned} \tag{7.5}$$

Altogether there are here three "dependent" currents $i^{d'}$, $i^{f'}$, and $i^{c'}$, each depending upon the other two only. The fact that the remaining two currents $i^{a'}$ and $i^{b'}$ remain independent may be expressed by the two equations of independence

$$\begin{aligned} i^{a'} &= i^{a'} \\ i^{b'} &= i^{b'} \end{aligned} \tag{7.6}$$

Hence, the actual five branch currents $i^{a'}$, $i^{b'}$, $i^{c'}$, $i^{d'}$, and $i^{f'}$ may be expressed in terms of the two independent branch currents $i^{a'}$ and $i^{b'}$ by the equation $\mathbf{i}_1 = \mathbf{C}' \cdot \mathbf{i}_2$.

$$\begin{aligned} i^{a'} &= \quad i^{a'} \\ i^{b'} &= \qquad\quad + i^{b'} \\ i^{c'} &= -\, i^{a'} - i^{b'} \\ i^{d'} &= \quad i^{a'} + i^{b'} \\ i^{f'} &= \qquad\quad - i^{b'} \end{aligned} \qquad \mathbf{C}' = \begin{array}{c|cc|} & a' & b' \\ \hline a' & 1 & \\ b' & & 1 \\ c' & -1 & -1 \\ d' & 1 & 1 \\ f' & & -1 \\ \hline \end{array} \qquad \mathbf{C} = \begin{array}{c|cc|} & a' & b' \\ \hline a & 1 & \\ b & & 1 \\ c & -1 & -1 \\ d & 1 & 1 \\ f & & -1 \\ \hline \end{array} \tag{7.7}$$

$$\mathbf{i}_1 = \begin{array}{|c|c|c|c|c|} a' & b' & c' & d' & f' \\ \hline i^{a'} & i^{b'} & i^{c'} & i^{d'} & i^{f'} \\ \hline \end{array} \qquad \mathbf{i}_2 = \begin{array}{|c|c|} a' & b' \\ \hline i^{a'} & i^{b'} \\ \hline \end{array}$$

(*c*) The set of equations 7.7 represents a relation between all the currents (dependent and independent) flowing in the individual coils and between the two *independent* currents. The $\mathbf{C}'$ matrix has exactly the same form as when the left-hand currents represent *another* set of ~~six~~ *independent* currents $i^a \cdots i^f$ flowing in the ~~six~~ meshes of the primitive network of Fig. 7.3*a* (that is, in the unconstrained system). Hence when the primes are removed on the left-hand side to change $\mathbf{i}_1 = \mathbf{C}' \cdot \mathbf{i}_2$ into $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$, an unconstrained primitive system is automatically introduced, whose equation of performance is easy to establish.

The difference between $\mathbf{C}$ and $\mathbf{C}'$ should be noted. Although both have the same components, the left-hand indices in $\mathbf{C}'$ are primed (referring to the branches of the interconnected network) while in $\mathbf{C}$ they are unprimed (referring to the meshes of the primitive network).

It should be noted that the *primitive network with six meshes, the unconstrained system, possesses a set of six differential equations* $\mathbf{e} = \mathbf{Z}\cdot\mathbf{i}$ *with six independent variables*, that are to be transformed with the aid of $\mathbf{C}$ to the two equations $\mathbf{e}' = \mathbf{Z}'\cdot\mathbf{i}'$ of the constrained network. When, however, the six currents $i^{a'} \cdots i^{f'}$ are considered branch currents and are partly dependent and partly independent, there cannot be associated with them a set of six differential equations $\mathbf{e} = \mathbf{Z}\cdot\mathbf{i}$ with six independent variables.*

(*d*) Hence, *in interconnecting individual coils into networks, the transformation* $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$ *may also be looked upon as a relation* $\mathbf{i}_1 = \mathbf{C}'\cdot\mathbf{i}_2$ *between the branch currents.* It consists of two sets of equations: (1) the P equations of constraint $\mathbf{B}\cdot\mathbf{i} = 0$ *rearranged* so that only independent currents occur on the right-hand side; (2) as many equations of independence as there are independent or "new" variables (some of these independent variables may change signs, of course), namely M.

When, however, not individual coils but whole networks are interconnected into larger networks, the above simple relation does not hold true and the $\mathbf{C}$ *tensor cannot be said to represent a relation between the branch currents.*

Of course it is easier to establish $\mathbf{C}$ (representing the interconnection of coils) without the intermediary step of equations of constraint, but cases will be encountered (such as the method of symmetrical components) where the interconnection of coils should at first be represented in the form of equations of constraint instead of $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$.

Steps in Expressing $\mathbf{B}\cdot\mathbf{i} = 0$ as $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$

Hence, a set of equations of constraint $\mathbf{B}\cdot\mathbf{i} = 0$ may be expressed as $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$ by the following steps:

1. In each equation of constraint express one (any one) of the currents in terms of the others (that is, carry one of the currents to the left-hand side, all the others to the right-hand side of the equation). This

* It is shown in *T.A.N.*, Chapter XVI, that, if the network is looked upon as an "orthogonal" network with an equation of performance $\mathbf{e} + \mathbf{E} = \mathbf{z}\cdot(\mathbf{i} + \mathbf{I})$, then the currents in the coils of the primitive network $i^a \cdots i^f$ are numerically equal to the currents $i^{a'} \cdots i^{f'}$ in the coils of the given network. That is, when the "dual" axes are also considered, then the branches do possess the same set of equations that the primitive system does. That set, however, is not $\mathbf{e} = \mathbf{z}\cdot\mathbf{i}$ but $\mathbf{e} + \mathbf{E} = \mathbf{z}\cdot(\mathbf{i} + \mathbf{I})$.

step defines as many "dependent" currents as there are equations of constraint.

2. By substitution, adjust the equation so that on the right-hand side only the independent currents occur.

These two steps give as many equations of the needed $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$ as there are equations of constraint available. Or rather, they still are the equations of constraint but *rearranged* in a more suitable form.

3. Equate to themselves the independent currrens. This step gives the remaining equations of $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$.

4. If the primes are removed, the collection of the two sets of equations (the rearranged equations of constraint and the equations of independence) is the required $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$, setting up a relation between the equations of the unconstrained and the constrained system.

Example of Changing $\mathbf{B}\cdot\mathbf{i} = 0$ to $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$

(*a*) For instance, in the case of two transformers (Fig. 7.4), let the equations of constraint $\mathbf{B}\cdot\mathbf{i} = 0$ represent the assumption that the sum of the m.m.f.'s around each closed magnetic circuit (the so-called "magnetizing" currents i^m) is zero. That is, let

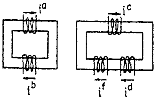


FIG. 7.4.

$$\begin{aligned} n_a i^a + n_b i^b &= i^m = 0 \\ n_c i^c + n_d i^d + n_f i^f &= i^n = 0 \end{aligned} \qquad 7.8$$

1. There are five currents i^a, i^b, i^c, i^d, and i^f. Assume arbitrarily that i^a and i^c are dependent currents (hence i^b, i^d, and i^f are independent.)

$$\begin{aligned} i^a &= -\frac{n_b}{n_a} i^b \\ i^c &= -\frac{n_d}{n_c} i^d - \frac{n_f}{n_c} i^f \end{aligned} \qquad 7.9$$

2. There is no need to readjust the equations since on the right-hand side only the independent currents occur.

3. Equating the independent currents to each other

$$i^b = i^b \qquad i^d = i^d \qquad i^f = i^f \qquad 7.10$$

4. By placing primes to the currents on the right-hand side, the five independent currents are changed to three dependent currents by $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$.

$$
\begin{aligned}
i^a &= -(n_b/n_a)i^{b'} \\
i^b &= i^{b'} \\
i^c &= -(n_d/n_c)i^{d'} - (n_f/n_c)i^{f'} \\
i^d &= i^{d'} \\
i^f &= i^{f'}
\end{aligned}
\qquad
\mathbf{C} = \begin{array}{c|ccc|}
 & \mathbf{b}' & \mathbf{d}' & \mathbf{f}' \\
\hline
\mathbf{a} & -n_b/n_a & & \\
\mathbf{b} & 1 & & \\
\mathbf{c} & & -n_d/n_c & -n_f/n_c \\
\mathbf{d} & & 1 & \\
\mathbf{f} & & & 1 \\
\hline
\end{array}
\tag{7.11}
$$

(*b*) Putting primes to the currents on the right-hand side is equivalent to introducing a hypothetical network, the "constrained" network in which the independent currents $\mathbf{i}'$ flow, just as in removing the primes from the currents on the left-hand side of equation 7.7 was equivalent to introducing a hypothetical network, the "unconstrained" primitive network, in which $\mathbf{i}$ flows. *The removal or addition of primes introduces a new set of variables and thereby it signifies the creation of a new network.*

The creation of two networks with primed and unprimed variables is equivalent to the creation of two sets of equations $\mathbf{e} = \mathbf{Z}\cdot\mathbf{i}$ *and* $\mathbf{e}' = \mathbf{Z}'\cdot\mathbf{i}'$ *that may now be transformed into each other with the aid of* $\mathbf{C}$.

(*c*) These new currents $i^{b'}$, $i^{d'}$, and $i^{f'}$ are not equal to the actual currents flowing in coils Z_{bb}, Z_{dd}, and Z_{ff} but are only approximations to them. They are *hypothetical* currents, the so-called "load" currents.

It is possible to say that: (1) Before neglecting the magnetizing currents, the reference frame of the unconstrained system consists of the *five* meshes **a, b, c, d,** and **f**; (2) after neglecting the magnetizing current, the reference frame of the constrained system consists of *three* meshes only, **b′, d′,** and **f′**.

Steps in Expressing $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$ as $\mathbf{B}\cdot\mathbf{i} = 0$

(*a*) The reverse problem sometimes arises, to establish the equation of constraint $\mathbf{B}\cdot\mathbf{i} = 0$ if $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$ is known. In simple cases it is only a question of picking out and removing the "equations of independence." Hence:

1. Denote the new currents, using the prime convention.
2. Pick out the equations of independence such as $i^a = i^{a'}$, $i^b = i^{b'}$, etc.
3. If one of the equations of independence has the form $i^b = -i^{b'}$,

this necessitates *multiplying every term in the column f′ by −1.* (This step reverses the direction of $i^{h'}$ to agree with i^h.)

4. The remaining equations are the equations of constraint. If needed, they may be brought to the form $\mathbf{B}\cdot\mathbf{i} = 0$ or $\mathbf{B}\cdot\mathbf{i}' = 0$.

(*b*) Expressed in another way, **B** is found from **C** as follows:

1. Rearrange **C** as a compound tensor

$$\mathbf{C} = \begin{array}{|c|}\hline \mathbf{I} \\ \hline \mathbf{C}' \\ \hline\end{array} \tag{7.12}$$

(where **I** is the unit tensor) by writing first the equations of independence.

2. Subtract **I**, where **I** has as many rows as **C** has, so that

$$\mathbf{B} = \mathbf{C} - \mathbf{I} \tag{7.13}$$

$$= \begin{array}{|c|c|}\hline \mathbf{I} & \\ \hline \mathbf{C}' & \\ \hline\end{array} - \begin{array}{|c|c|}\hline \mathbf{I} & \\ \hline & \mathbf{I} \\ \hline\end{array} = \begin{array}{|c|c|}\hline & \\ \hline \mathbf{C}' & -\mathbf{I} \\ \hline\end{array} = \mathbf{B} \tag{7.14}$$

Hence

$$\mathbf{B} = \mathbf{C}' - \mathbf{I} \tag{7.15}$$

EXERCISES

1. What are the equations of constraint $\mathbf{B}\cdot\mathbf{i} = 0$ of Fig. 7.5?
2. Change $\mathbf{B}\cdot\mathbf{i} = 0$ to $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$.

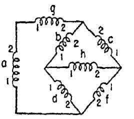


FIG. 7.5.

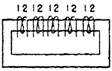


FIG. 7.6.

3. What is the equation of constraint of the five-winding transformer of Fig. 7.6?
4. Set up $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$ for Fig. 7.6 that neglects the magnetizing current.

CHAPTER 8

UNBALANCED MULTIWINDING TRANSFORMERS*

The Method of Analysis

The analysis of multiwinding transformers differs in two respects from that of the circuits hitherto considered.

1. Since the magnetizing current in each closed magnetic circuit may be neglected, the number of new variables $\mathbf{i}'$ in such cases is *less* than the number of meshes by as many as there are closed magnetic circuits. That is, the actual mesh network is replaced by a hypothetical network with fewer meshes.

2. In place of the large number of usual self and mutual reactances, hypothetical "bucking" reactances are generally used, whose number is less. That is, the actual coils are replaced by hypothetical coils possessing different types of self and mutual inductances.

3. In balanced three-phase systems the number of variables may be reduced to a third of those in the unbalanced case.

Successive Transformations $\mathbf{C}_1 \cdot \mathbf{C}_2$

The analysis automatically divides into two steps.

1. The step of interconnecting the coils is represented by $\mathbf{C}_1$. That is, first establish $\mathbf{i} = \mathbf{C}_1 \cdot \mathbf{i}'$.

2. The step of neglecting the magnetizing current is represented by $\mathbf{C}_2$. That is, establish $\mathbf{i}' = \mathbf{C}_2 \cdot \mathbf{i}''$.

Their product

$$\mathbf{C} = \mathbf{C}_1 \cdot \mathbf{C}_2 \qquad 8.1$$

performs the two analytical operations in one step, changing $\mathbf{i}$ to $\mathbf{i}''$ by $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}''$, so that $\mathbf{Z}' = \mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C}$, etc., gives the final results.

The Steps to Establish $\mathbf{C}_2$

In establishing $\mathbf{C}_2$ (after $\mathbf{C}_1$ has already been found), the following steps are performed.

1. Set up the equations of constraint $\mathbf{B} \cdot \mathbf{i} = 0$ of the magnetic circuit *before the coils are interconnected*, since in that case the equations are easily written.

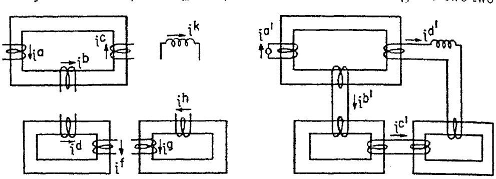
* *T.A.N.*, p. 280.

2. Replace in these equations with the aid of the already established $\mathbf{i} = \mathbf{C}_1 \cdot \mathbf{i}'$ the currents of the primitive network with the currents of the actual network by simple substitution or by $\mathbf{B}' = \mathbf{B} \cdot \mathbf{C}$. This step gives the equations of constraint $\mathbf{B}' \cdot \mathbf{i}' = 0$ in terms of the currents of the actual network.

3. Express these equations of constraint as $\mathbf{i}' = \mathbf{C}_2 \cdot \mathbf{i}''$ by the standard steps, giving $\mathbf{C}_2$.

Example of a Load-Ratio Control System

As an example, let the $\mathbf{C}$ of Fig. 8.1*b* be established (a load-ratio control system with regulating unit) where one three-winding and two two-



(*a*) Primitive system. (*b*) Interconnected system.

FIG. 8.1. Load-ratio control.

winding transformers, also a load, are interconnected into a four-mesh network.

Its primitive network, shown in Fig. 8.1*a*, has eight meshes and eight currents $i^a \cdots i^k$; the given network has four meshes and four independent currents $i^{a'}, i^{b'}, i^{c'}, i^{d'}$.

1. Equating the old currents and new currents flowing in each coil, the equations $\mathbf{i} = \mathbf{C}_1 \cdot \mathbf{i}'$ are

$$\begin{aligned}
i^a &= i^{a'} \\
i^b &= i^{b'} \\
i^c &= i^{d'} \\
i^d &= -i^{b'} \\
i^f &= -i^{c'} \\
i^g &= +i^{c'} \\
i^h &= i^{d'} \\
i^k &= i^{d'}
\end{aligned}
\qquad
\mathbf{C}_1 = \begin{array}{c|c|c|c|c|}
 & a' & b' & c' & d' \\
\hline
a & 1 & & & \\
b & & 1 & & \\
c & & & & 1 \\
d & & -1 & & \\
f & & & -1 & \\
g & & & 1 & \\
h & & & & 1 \\
k & & & & 1 \\
\hline
\end{array} \qquad 8.2$$

The coefficients of the new currents give $\mathbf{C}_1$ that changes *eight* variables into *four* variables.

If the magnetizing currents are not to be neglected, then $\mathbf{C}_{1t}\cdot\mathbf{Z}\cdot\mathbf{C}_1$ would give $\mathbf{Z}'$, and so on.

2. Neglecting the magnetizing currents of the three transformers *before* the coils are interconnected (Fig. 8.1a), the three equations of constraint $\mathbf{B}\cdot\mathbf{i}$ are in terms of $\mathbf{i}$ (in terms of five old currents).

$$\begin{aligned} n_a i^a + n_b i^b + n_c i^c &= 0 \\ n_d i^d + n_f i^f &= 0 \\ n_g i^g + n_h i^h &= 0 \end{aligned} \qquad 8.3$$

3. Replacing the old currents by the new currents with the aid of equation 8.2 (that is, replacing i^d by $-i^{c'}$, etc.) or by $\mathbf{B}\cdot\mathbf{C}_1 = \mathbf{B}'$, the three equations of constraint $\mathbf{B}'\cdot\mathbf{i}' = \mathbf{0}$ in terms of $\mathbf{i}'$ (in terms of four new currents) are

$$\begin{aligned} n_a i^{a'} + n_b i^{b'} + n_c i^{d'} &= 0 \\ -\, n_d i^{b'} - n_f i^{c'} &= 0 \\ n_g i^{c'} + n_h i^{d'} &= 0 \end{aligned} \qquad 8.4$$

4. Three of the currents, say $i^{a'}$, $i^{b'}$, and $i^{c'}$, may be expressed in terms of the remaining fourth current $i^{d'}$ as

$$\begin{aligned} i^a &= -\frac{n_b}{n_a} i^{b'} - \frac{n_c}{n_a} i^{d'} \\ i^{b'} &= -\frac{n_f}{n_d} i^{c'} \\ i^{c'} &= -\frac{n_h}{n_g} i^{d'} \end{aligned} \qquad 8.5$$

Or adjusting the right-hand side so that it should contain only $i^{d'}$ (by replacing $i^{b'}$ and $i^{c'}$ on the right-hand side by their values from the second and third equations)

$$\begin{aligned} i^{a'} &= -\left(\frac{n_b}{n_a}\frac{n_f}{n_d}\frac{n_h}{n_g} + \frac{n_c}{n_a}\right) i^{d'} &&= N_1 i^{d'} \\ i^{b'} &= \frac{n_f}{n_d}\frac{n_h}{n_g} i^{d'} &&= N_2 i^{d'} \\ i^{c'} &= -\frac{n_h}{n_g} i^{d'} &&= N_3 i^{d'} \end{aligned} \qquad 8.6$$

Now three of the currents are expressed in terms of the fourth current.

5. Equating the remaining current $i^{d'}$ with itself, the three equations of constraint, equation 8.6, may be expressed as a transformation $\mathbf{i}' = \mathbf{C}_2\cdot\mathbf{i}''$ between the actual network with currents $\mathbf{i}'$ and a hypothetical network with currents $\mathbf{i}''$

$$\begin{aligned} i^{a'} &= N_1 i^{d''} \\ i^{b'} &= N_2 i^{d''} \\ i^{c'} &= N_3 i^{d''} \\ i^{d'} &= i^{d''} \end{aligned} \qquad \mathbf{C}_2 = \begin{array}{c|c|} & d'' \\ \hline a' & N_1 \\ b' & N_2 \\ c' & N_3 \\ d' & 1 \\ \hline \end{array} \tag{8.7}$$

$\mathbf{C}_2$ changes *four* variables into *one* variable. Hence, the effect of neglecting magnetizing currents is to reduce the number of variables by as many as there are closed magnetic circuits.

6. The product of $\mathbf{C}_1$ and $\mathbf{C}_2$ is

$$\mathbf{C} = \mathbf{C}_1\cdot\mathbf{C}_2 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & -1 & & \\ & & -1 & \\ & & 1 & \\ & & & 1 \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ 1 \end{bmatrix} = \begin{array}{c|c|} & d'' \\ \hline a & N_1 \\ b & N_2 \\ c & 1 \\ \hline d & -N_2 \\ f & -N_3 \\ \hline g & N_3 \\ h & 1 \\ \hline k & 1 \\ \hline \end{array} = \begin{bmatrix} \mathbf{C}_a \\ \mathbf{C}_b \\ \mathbf{C}_c \\ \mathbf{C}_d \end{bmatrix} \tag{8.8}$$

$\mathbf{C}$ changes *eight* variables into *one* variable.

If $\mathbf{Z}$ of the primitive network (Fig. 8.1*a*) is given in terms of *actual* reactances, also $\mathbf{e}$, then $\mathbf{Z}' = \mathbf{C}_t\cdot\mathbf{Z}\cdot\mathbf{C}$, $\mathbf{e}' = \mathbf{C}_t\cdot\mathbf{e}$, $\mathbf{i}' = \mathbf{Z}'^{-1}\cdot\mathbf{e}$. The currents in the individual coils are $\mathbf{i}_c = \mathbf{C}\cdot\mathbf{i}'$ and differences of potentials $\mathbf{e}_c = \mathbf{Z}\cdot\mathbf{C}\cdot\mathbf{i}'$. The load losses (not including the exciting loss) are the real part of $\mathbf{i}'^*\cdot\mathbf{e}' = \mathbf{i}'^*\cdot\mathbf{Z}'\cdot\mathbf{i}'$ or the real part of $\mathbf{i}_c^*\cdot\mathbf{Z}\cdot\mathbf{i}_c$. (See equation 9.1.) In place of actual reactances, however, it is customary to use a new type of reactance, the so-called bucking reactance.

Bucking Reactance

(*a*) Let **Z** of a two-winding transformer be

$$\mathbf{Z} = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & Z_{11} & Z_{12} \\ 2 & Z_{21} & Z_{22} \end{array} \qquad \text{where} \qquad \begin{aligned} Z_{11} &= n_1^2 z_{11} \\ Z_{12} &= n_1 n_2 z_{12} \end{aligned} \qquad Z_{22} = n_2^2 z_{22}$$

If the magnetizing current is neglected, then

$$n_1 i^1 + n_2 i^2 = 0 \qquad\qquad \mathbf{C} = \begin{array}{c|c} & 1 \\ \hline 1 & 1 \\ 2 & -\dfrac{n_1}{n_2} \end{array} \qquad 8.9$$

$$i^2 = -\frac{n_1}{n_2} i^1$$

and

$$\mathbf{Z}' = \mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C} = \begin{array}{c|c} & 1 \\ \hline 1 & \left(\dfrac{n_1}{n_2}\right)^2 Z_{11} - 2Z_{12}\dfrac{n_1}{n_2} + Z_{22} \end{array} = \begin{array}{c|c} & 1 \\ \hline 1 & Z'_{1-2} \end{array}$$

where

$$Z'_{1-2} = \left(\frac{n_1}{n_2}\right)^2 Z_{11} - 2Z_{12}\frac{n_1}{n_2} + Z_{22} = n_1 n_1 (z_{11} - 2z_{12} + z_{22}) \qquad 8.10$$

It should be noted that the original **Z** contains four different constants, while **Z′** contains only one, namely $\mathbf{Z}'_{1-2}$.

(*b*) The question arises: Instead of using four constants in the original **Z,** may it not be possible to use the single constant $\mathbf{Z}'_{1-2}$ as the component of **Z,** so that after the elimination of the magnetizing current **Z′** still has the same form as before?

By trial and error it is found that if **Z** is written as

$$\mathbf{Z} = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0 & -\dfrac{1}{2}\dfrac{n_2}{n_1} Z'_{1-2} \\ 2 & -\dfrac{1}{2}\dfrac{n_2}{n_1} Z'_{1-2} & 0 \end{array} = \begin{array}{|cc|} 0 & Z_{1-2} \\ Z_{1-2} & 0 \end{array} \qquad 8.11$$

in that case $\mathbf{Z}' = \mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C} = Z'_{1-2}$.

The impedance

$$Z_{1-2} = -\frac{1}{2}\frac{n_2}{n_1}Z'_{1-2} = -\frac{1}{2}n_1n_2(z_{11} - 2z_{12} + z_{22}) \qquad 8.12$$

is to be called the "unreferred bucking impedance" since it is not referred to any reference winding. In general,

$$Z_{2-3} = -\frac{1}{2}\frac{n_2n_3}{n_1n_1}Z'_{2-3} \qquad 8.13$$

if a third winding is the reference winding in defining Z'_{2-3}.

(*c*) In the primitive three-winding transformer it is again found that if the unreferred bucking impedances are arranged in the form of a matrix with zero diagonals, as

$$\mathbf{Z} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & Z_{1-2} & Z_{1-3} \\ 2 & Z_{1-2} & 0 & Z_{2-3} \\ 3 & Z_{1-3} & Z_{2-3} & 0 \end{array} \qquad 8.14$$

then the same answer is found after the magnetizing current is eliminated as when the usual self and mutual impedances are used.

(*d*) This process of replacing actual impedances by bucking impedances is equivalent to introducing *hypothetical* coils whose self-impedance is zero, but whose mutual impedances are the unreferred bucking impedances.

The method of establishing $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *and the calculation of currents, etc., remain unchanged whether "actual" or "unreferred bucking" impedances are used.*

Transformation to Change Z'_{1-2} to Z_{1-2}

(*a*) In multiwinding transformers usually the bucking impedances Z'_{n-m} referred to a particular winding are known. In that case, in $\mathbf{Z}$ of the primitive transformer, the given Z'_{n-m} may be still arranged as the Z_{n-m}

$$\mathbf{Z}_0 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & Z'_{1-2} & Z'_{1-3} \\ 2 & Z'_{1-2} & 0 & Z'_{2-3} \\ 3 & Z'_{1-3} & Z'_{2-3} & 0 \end{array} \qquad 8.15$$

but then a transformation tensor $\mathbf{C}_0$ has to be established that changes the referred to unreferred bucking impedances. This $\mathbf{C}_0$ has the form (when winding 1 is the reference winding)

$$\mathbf{C}_0 = -\tfrac{1}{2}\ \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & n_1/n_1 & & \\ 2 & & n_2/n_1 & \\ 3 & & & n_3/n_1 \end{array} \qquad 8.16$$

so that $\mathbf{C}_{0t}\cdot\mathbf{Z}_0\cdot\mathbf{C}_0$ represents a $\mathbf{Z}$ containing only Z_{n-m}, namely, equation 8.14.

(*b*) Instead of establishing first $\mathbf{C}_{0t}\cdot\mathbf{Z}_0\cdot\mathbf{C}_0$, it is possible to start with $\mathbf{Z}_0$, then employ

$$\mathbf{C} = \mathbf{C}_0\cdot\mathbf{C}_1\cdot\mathbf{C}_2 \qquad 8.17$$

to find $\mathbf{Z}'$ where:

1. $\mathbf{C}_0$ changes the referred to unreferred bucking impedances (that is, it changes $\mathbf{Z}_0$ to $\mathbf{Z}$).
2. $\mathbf{C}_1$ interconnects coils.
3. $\mathbf{C}_2$ neglects magnetizing currents.

$\mathbf{Z}' = \mathbf{C}_t\cdot\mathbf{Z}_0\cdot\mathbf{C}$ gives the so-called load mutual impedances expressed in terms of bucking impedances.

(*c*) In finding $\mathbf{e}' = \mathbf{C}_t\cdot\mathbf{e}$ of the transformation, $\mathbf{C}$ is still defined as $\mathbf{C} = \mathbf{C}_1\cdot\mathbf{C}_2$ since $\mathbf{C}_0$ is used only to bring $\mathbf{Z}$ to its correct form.

(*d*) The load losses may be found by $\mathbf{e}'^{*}\cdot\mathbf{i}'$ or by $\mathbf{i}_c^{*}\cdot\mathbf{Z}\cdot\mathbf{i}_c$. Not only the currents but also the losses are the same whether standard self and mutual impedances or bucking impedances are used in $\mathbf{Z}$ of the primitive system.

(*e*) When the coils of a multiwinding transformer are not interconnected, $\mathbf{C}_1$ becomes a unit tensor. The above formulas without any change give the "load mutual impedances," etc., of the transformer.

(*f*) In practical work the leakage impedances Z'_{j-k} are given in "per unit." *In that case all windings are assumed to have the same number of turns* ($n_1 = n_2 = n_3 = 1$) *and* $\mathbf{C}_0$ *degenerates into a scalar* $-\tfrac{1}{2}$.

Short-Circuit Calculation of a Four-Winding Transformer*

Find the currents in a four-winding transformer when the second winding is short-circuited and the voltages on the other windings are

* Blume, *Transformer Engineering*, John Wiley & Sons.

maintained. The per unit unreferred bucking reactances of the four coils are given as

$Z'_{1-2} = Z'_{2-1} = 3.95$ per cent

$Z'_{1-3} = Z'_{3-1} = 13.3$ per cent

$Z_{1-4} = Z_{4-1} = 23$ per cent

$Z_{2-3} = Z_{3-2} = 10.6$ per cent

$Z_{2-4} = Z_{4-2} = 19$ per cent

$Z_{3-4} = Z_{4-3} = 8.7$ per cent

$$\mathbf{Z}_0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} & 0.0395 & 0.133 & 0.23 \\ 0.0395 & & 0.106 & 0.19 \\ 0.133 & 0.106 & & 0.087 \\ 0.23 & 0.19 & 0.087 & \end{array}\right] \end{array} \qquad \mathbf{e} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 1 & 1 \end{array}\right] \end{array} \tag{8.18}$$

If the magnetizing current in winding 4 is neglected, than

$$\mathbf{C}_0 = -\tfrac{1}{2};\ \mathbf{C}_1 = \mathbf{I};\ \mathbf{C}_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} 1' & 2' & 3' \end{array} \\ \left[\begin{array}{ccc} 1 & & \\ & 1 & \\ & & 1 \\ -1 & -1 & -1 \end{array}\right] \end{array} \qquad \mathbf{C} = \mathbf{C}_0 \cdot \mathbf{C}_1 \cdot \mathbf{C}_2 \tag{8.19}$$

$$\mathbf{e}' = (\mathbf{C}_1 \cdot \mathbf{C}_2)_t \cdot \mathbf{e} = \begin{array}{c} 1' \\ 2' \\ 3' \end{array} \left[\begin{array}{c} 0 \\ -1 \\ 0 \end{array}\right] \qquad \mathbf{Z}' = \mathbf{C}_t \cdot \mathbf{Z}_0 \cdot \mathbf{C} = \begin{array}{c} \\ 1' \\ 2' \\ 3' \end{array} \begin{array}{c} \begin{array}{ccc} 1' & 2' & 3' \end{array} \\ \left[\begin{array}{ccc} 0.23 & 0.19 & 0.092 \\ 0.19 & 0.19 & 0.085 \\ 0.092 & 0.085 & 0.087 \end{array}\right] \end{array} \tag{8.20}$$

$$\mathbf{Y}' = \begin{array}{c} \\ 1' \\ 2' \\ 3' \end{array} \begin{array}{c} \begin{array}{ccc} 1' & 2' & 3' \end{array} \\ \left[\begin{array}{ccc} 5.95 & -4.59 & 0.34 \\ -4.59 & 6.10 & -0.49 \\ -0.34 & 0.49 & 1.84 \end{array}\right] \end{array} \qquad \mathbf{i}' = \mathbf{Y}' \cdot \mathbf{e}' = \begin{array}{c} 1' \\ 2' \\ 3' \end{array} \left[\begin{array}{c} 4.59 \\ -6.1 \\ 0.49 \end{array}\right] \tag{8.21}$$

From $\mathbf{i}_c = \mathbf{C} \cdot \mathbf{i}'$, $i^4 = 1.02$.

Load-Ratio Control System

If in the load-ratio control system, Fig. 8.1*a*, $\mathbf{Z}$ is given in terms of unreferred bucking impedances, then

$$
Z = \begin{array}{c|cccccccc|}
 & a & b & c & d & f & g & h & k \\
\hline
a & & Z_{a-b} & Z_{a-c} & & & & & \\
b & Z_{a-b} & & Z_{b-c} & & & & & \\
c & Z_{a-c} & Z_{b-c} & & & & & & \\
d & & & & & Z_{d-f} & & & \\
f & & & & Z_{d-f} & & & & \\
g & & & & & & & Z_{g-h} & \\
h & & & & & & Z_{g-h} & & \\
k & & & & & & & & Z_{kk} \\
\hline
\end{array} \qquad 8.22
$$

Because of the smaller number of constants (six instead of thirteen) and the greater number of zero terms, the calculation and the results are simpler.

C is given in equation 8·8.

$$
\mathbf{C}_t\cdot\mathbf{Z}\cdot\mathbf{C} = \mathbf{Z}' = \begin{array}{c|c|}
 & d'' \\
\hline
d'' & \begin{matrix} 2Z_{a-b}N_1N_2 + 2Z_{a-c}N_1 + 2Z_{b-c}N_2 \\ + 2Z_{d-f}N_2N_3 + 2Z_{g-h}N_3 + Z_{kk} \end{matrix} \\
\hline
\end{array} = Z'
$$

$$
\mathbf{e}' = \mathbf{C}_t\cdot\mathbf{e} = \begin{array}{|c|}
d'' \\
\hline
N_1 e \\
\hline
\end{array} \qquad \mathbf{i}' = \mathbf{Z}'^{-1}\cdot\mathbf{e} = \frac{N_1 e}{Z'} = \begin{array}{|c|}
d'' \\
\hline
i^{d''} \\
\hline
\end{array} \qquad 8.23
$$

The currents flowing in each coil are

$$
\mathbf{i}_c = \mathbf{C}\cdot\mathbf{i}' = \begin{array}{|c|c|c|c|c|c|c|c|}
a & b & c & d & f & g & h & k \\
\hline
N_1 i^{d''} & N_2 i^{d''} & i^{d''} & -N_2 i^{d''} & -N_3 i^{d''} & N_3 i^{d''} & i^{d''} & i^{d''} \\
\hline
\end{array}
$$

The differences of potential appearing across each of the eight coils is $\mathbf{Z}\cdot\mathbf{C}\cdot\mathbf{i}' = \mathbf{e}_c =$

$$
\begin{array}{|c|c|c|c|c|c|c|c|}
a & b & c & d & f & g & h & k \\
\hline
\begin{matrix}(Z_{a-b}N_2 \\ + Z_{a-c}) \\ i^{d''}\end{matrix} & \begin{matrix}(Z_{a-b}N_1 \\ + Z_{b-c}) \\ i^{d''}\end{matrix} & \begin{matrix}(Z_{a-c}N_1 \\ + Z_{b-c}) \\ i^{d''}\end{matrix} & -Z_{d-f}N_3 i^{d''} & -Z_{d-f}N_3 i^{d''} & Z_{g-h} i^{d''} & Z_{g-h}N_3 i^{d''} & Z_{kk} i^{d''} \\
\hline
\end{array}
$$

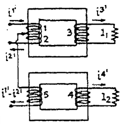
TABLE II

UNBALANCED TRANSFORMER CONNECTIONS AND THEIR TRANSFORMATION MATRICES

First column—C_1 shows interconnection of coils.

Second column—C_2 neglects magnetizing currents.

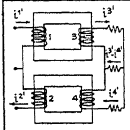
Third column—C represents their resultant.



$$C_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 1_1 \\ 1_2 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1' & 2' & 3' & 4' \end{array} \\ \left[\begin{array}{cccc} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & -1 & & \\ & & 1 & \\ & & & 1 \end{array}\right] \end{array} \qquad C_2 = \begin{array}{c} \\ 1' \\ 2' \\ 3' \\ 4' \end{array}\!\!\begin{array}{c} \begin{array}{cc} 3'' & 4'' \end{array} \\ \left[\begin{array}{cc} -N_1 & -N_2 \\ -N_1 & -N_3 \\ 1 & \\ & 1 \end{array}\right] \end{array} \qquad C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 1_1 \\ 1_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 3'' & 4'' \end{array} \\ \left[\begin{array}{cc} -N_1 & -N_2 \\ -N_1 & -N_3 \\ 1 & \\ & 1 \\ & -N_2+N_3 \\ 1 & \\ & 1 \end{array}\right] \end{array}$$

$$N_1 = \frac{n_3}{n_1+n_2};\quad N_2 = \frac{n_2 n_4}{n_5(n_1+n_2)};\quad N_3 = \frac{-n_1 n_4}{n_5(n_1+n_2)}$$

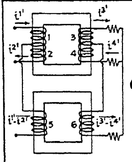
SCOTT CONNECTION $3\phi \rightarrow 2\phi$



$$C_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 1_1 \\ 1_2 \\ 1_3 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1' & 2' & 3' & 4' \end{array} \\ \left[\begin{array}{cccc} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ & & 1 & \\ & & & 1 \\ & & 1 & -1 \end{array}\right] \end{array} \qquad C_2 = \begin{array}{c} \\ 1' \\ 2' \\ 3' \\ 4' \end{array}\!\!\begin{array}{c} \begin{array}{cc} 3'' & 4'' \end{array} \\ \left[\begin{array}{cc} -N_1 & \\ & -N_2 \\ 1 & \\ & 1 \end{array}\right] \end{array} \qquad C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 1_1 \\ 1_2 \\ 1_3 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 3'' & 4'' \end{array} \\ \left[\begin{array}{cc} -N_1 & \\ & -N_2 \\ 1 & \\ & 1 \\ 1 & \\ & 1 \\ 1 & -1 \end{array}\right] \end{array}$$

$$N_1 = \frac{n_3}{n_1};\quad N_2 = \frac{n_4}{n_2}$$

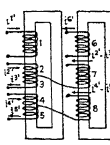
VEE-VEE CONNECTION



$$C_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 1_1 \\ 1_2 \\ 1_3 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1' & 2' & 3' & 4' \end{array} \\ \left[\begin{array}{cccc} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & -1 & & \\ & & 1 & -1 \\ & & 1 & \\ & & & 1 \\ & & 1 & -1 \end{array}\right] \end{array} \qquad C_2 = \begin{array}{c} \\ 1' \\ 2' \\ 3' \\ 4' \end{array}\!\!\begin{array}{c} \begin{array}{cc} 3'' & 4'' \end{array} \\ \left[\begin{array}{cc} -N_1 & -N_2 \\ -N_3 & -N_4 \\ 1 & \\ & 1 \end{array}\right] \end{array} \qquad C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 1_1 \\ 1_2 \\ 1_3 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 3'' & 4'' \end{array} \\ \left[\begin{array}{cc} -N_1 & -N_2 \\ -N_3 & -N_4 \\ 1 & \\ & 1 \\ -N_1+N_3 & -N_2+N_4 \\ 1 & -1 \\ 1 & \\ & 1 \\ 1 & -1 \end{array}\right] \end{array}$$

$$N_1 = (n_2 n_6 + n_3 n_5)/D$$
$$N_2 = (n_2 n_6 - n_4 n_5)/D$$
$$N_3 = (n_3 n_5 - n_1 n_6)/D$$
$$N_4 = (n_4 n_5 + n_1 n_6)/D$$
$$D = n_5(n_1+n_2)$$

TEE-TEE CONNECTION



$$C_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} 1' & 2' & 3' & 4' & 5' & 6' \end{array} \\ \left[\begin{array}{cccccc} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ & 1 & -1 & & & \\ & & & 1 & -1 & \end{array}\right] \end{array} \qquad C_2 = \begin{array}{c} \\ 1' \\ 2' \\ 3' \\ 4' \\ 5' \\ 6' \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 2'' & 3'' & 4'' & 5'' \end{array} \\ \left[\begin{array}{cccc} -N_1 & -N_2 & -N_3 & -N_4 \\ 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ -N_5 & N_5 & -N_6 & N_6 \end{array}\right] \end{array} \qquad C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 2'' & 3'' & 4'' & 5'' \end{array} \\ \left[\begin{array}{cccc} -N_1 & -N_2 & -N_3 & -N_4 \\ 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ -N_5 & N_5 & -N_6 & N_6 \\ 1 & -1 & & \\ & & 1 & -1 \end{array}\right] \end{array}$$

$$N_1 = \frac{n_2}{n_1},\ N_2 = \frac{n_3}{n_1},\ N_3 = \frac{n_4}{n_1}$$
$$N_4 = \frac{n_5}{n_1},\ N_5 = \frac{n_7}{n_1},\ N_6 = \frac{n_8}{n_1}$$

DOUBLE-SCOTT CONNECTION $2\phi \rightarrow 3\phi$

EXERCISES

1. Find C_1, C_2, and C for the *forked auto-transformer* of Fig. 8.2.
2. Find its Z' in terms of bucking impedances Z_{a-b}, etc.
3. With an impressed voltage as shown, what are the differences of potentials

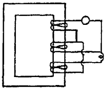


FIG. 8.2.

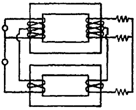


FIG. 8.3.

appearing across each coil of the forked auto-transformer?

4. Find C_1, C_2, and C for the two *tee-tee connected single-phase transformers* supplying an unbalanced three-phase load, as shown in Fig. 8.3.
5. Check C_1, C_2, and C and find Z, i, and e_c of the transformers shown in Table II.

CHAPTER 9

THE METHOD OF SYMMETRICAL COMPONENTS *

Conjugate Tensors

The method of symmetrical components introduces a group of transformations **C** whose components contain complex numbers. For that case the rules of tensor analysis assume a more general form.

(*a*) The conjugate of a complex number $A = a + jb$ is $a - jb$ and is denoted by an asterisk as A^*.

The conjugate of an n-matrix **Z** is denoted by **Z*** and is found by taking the conjugate of each of its elements.

$$\mathbf{Z} = \begin{array}{|c|c|c|}\hline 2+3j & 0 & 3 \\ \hline -5j & 5 & -3+2j \\ \hline j & -j & 0 \\ \hline \end{array} \qquad \mathbf{Z}^* = \begin{array}{|c|c|c|}\hline 2-3j & 0 & 3 \\ \hline 5j & 5 & -3-2j \\ \hline -j & j & 0 \\ \hline \end{array} \qquad 9.1$$

The conjugate of a tensor of valence n **A** is **A***, and it is found by taking the conjugate of its n-matrices in *every reference frame.*

(*b*) The following three rules should be noted

$$\begin{aligned} &1.\ (\mathbf{A}^*)^* = \mathbf{A} \\ &2.\ (\mathbf{A}\cdot\mathbf{B})^* = \mathbf{A}^*\cdot\mathbf{B}^* \qquad 9.2 \\ &3.\ (\mathbf{A}^{-1})^* = (\mathbf{A}^*)^{-1} \end{aligned}$$

(*c*) When the components of the vectors **e** and **i** are complex numbers, then the power is not **e·i** but

$$P = \mathbf{e}^*\cdot\mathbf{i} \qquad 9.3$$

Similarly a quadratic form is defined as

$$P = \mathbf{i}^*\cdot\mathbf{Z}\cdot\mathbf{i} \qquad 9.4$$

(*d*) When the components of the transformation tensor **C** contain complex numbers (as in the method of symmetrical components), then

* *T.A.N.*, Chapter XIII.

the laws of transformation of tensors differ in some respects from those given previously. In particular:

Whenever $\mathbf{C}_t$ *occurs in the law of transformation, it should be replaced by* $\mathbf{C}_t^*$. Hence

$$\boxed{\mathbf{e}' = \mathbf{C}_t^* \cdot \mathbf{e}} \qquad 9.5 \qquad\qquad \boxed{\mathbf{Z}' = \mathbf{C}_t^* \cdot \mathbf{Z} \cdot \mathbf{C}} \qquad 9.6$$

(Their proof is analogous to the previous laws except that $\mathbf{e} \cdot \mathbf{i}$ is replaced by $\mathbf{e}^* \cdot \mathbf{i}$.)

When $\mathbf{C}$ contains complex components, then "tensors" are often called "spinors," also "hermitian tensors."

The Hypothetical Reference Frame of Fortescue*

(*a*) Let three equal, symmetrically spaced and isolated coils (a primitive system with three coils) be given with two (not three) different mutual impedances between them (Fig. 9.1) such as occur in balanced induction and synchronous machines.

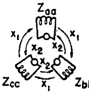


FIG. 9.1.

$$\mathbf{Z} = \begin{array}{c|ccc|} & \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \hline \mathrm{a} & Z & X_1 & X_2 \\ \mathrm{b} & X_2 & Z & X_1 \\ \mathrm{c} & X_1 & X_2 & Z \\ \hline \end{array}$$

To find the inverse of $\mathbf{Z}$, a determinant with three rows and columns has to be solved.

(*b*) Fortescue suggested replacing the three *actual* currents i^a, i^b, and i^c of the primitive system by three *hypothetical* currents i^0, i^1, and i^2 (zero-, positive-, and negative-sequence currents) with the formula $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$.

$$i^a = \frac{1}{\sqrt{3}}(i^0 + i^1 + i^2)$$

$$i^b = \frac{1}{\sqrt{3}}(i^0 + a^2 i^1 + a i^2)$$

$$i^c = \frac{1}{\sqrt{3}}(i^0 + a i^1 + a^2 i^2)$$

$$\mathbf{C} = \frac{1}{\sqrt{3}} \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline \mathrm{a} & 1 & 1 & 1 \\ \mathrm{b} & 1 & a^2 & a \\ \mathrm{c} & 1 & a & a^2 \\ \hline \end{array} \qquad 9.7$$

* *G.E.R.*, May, 1935; *T.A.N.*, Chapters XIII, XIX, and XX.

where

$$a = -\tfrac{1}{2} + j0.866 = \epsilon^{j120}$$

$$a^2 = -\tfrac{1}{2} - j0.866 = \epsilon^{-j120}$$

$$\text{Det of } \mathbf{C} = \sqrt{3}\,(a - a^2)$$

Also

$$1 + a + a^2 = 0$$

$$a^3 = 1\ ;\ a^4 = a$$

$$a^* = a^2;\ (a^2)^* = a$$

$$\mathbf{C}^{-1} = \frac{1}{\sqrt{3}}\ \begin{array}{c|ccc} & a & b & c \\ \hline 0 & 1 & 1 & 1 \\ 1 & 1 & a & a^2 \\ 2 & 1 & a^2 & a \end{array} \qquad 9.8$$

$$\mathbf{C}_t^* = \frac{1}{\sqrt{3}}\ \begin{array}{c|ccc} & a & b & c \\ \hline 0 & 1 & 1 & 1 \\ 1 & 1 & a & a^2 \\ 2 & 1 & a^2 & a \end{array} \qquad 9.9$$

(The factor $1/\sqrt{3}$ is introduced here to express the power in symmetrical components also by $\mathbf{e}^*\cdot\mathbf{i}$ instead of by $3\mathbf{e}^*\cdot\mathbf{i}$, as is usually done.)

(*c*) In the new hypothetical reference frame $\mathbf{Z}' = \mathbf{C}_t^*\cdot\mathbf{Z}\cdot\mathbf{C}$

$$\mathbf{Z}' = \tfrac{1}{3}\ \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & Z + X_1 + X_2 & & \\ 1 & & Z + a^2X_1 + aX_2 & \\ 2 & & & Z + aX_1 + a^2X_2 \end{array} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & Z_0 & & \\ 1 & & Z_1 & \\ 2 & & & Z_2 \end{array} \qquad 9.10$$

Hence in the new reference frame *the three coils have no mutual impedances* (their self-impedances are called zero-, positive-, and negative-sequence impedances), also $\mathbf{Z}'$ has only diagonal components, so that in finding $\mathbf{Z}^{-1}$ no determinant has to be solved.

(*d*) Expressed in another way, the method of symmetrical components replaces the actual coils of a network by hypothetical coils whose $\mathbf{Z}$ has several (and if possible only) diagonal components. Then the inverse calculation is simpler.

In addition to changing the coils of a network, the method of symmetrical components also replaces the *actual* given network by a *hypothetical* sequence network that in general contains *several independent subnetworks having no mutual impedance between them.* As a result the inverse calculation of $\mathbf{Z}'$ is simpler.

To find $\mathbf{Z}'$ of this hypothetical network is the purpose of the present study.

(*e*) In *two-phase* problems, Fortescue's transformations become (for the primitive system with two coils)

$$i^a = \tfrac{1}{2}(i^1 + i^2) \qquad\qquad \mathrm{C} = \tfrac{1}{2}\begin{array}{c|cc} & 1 & 2 \\ \hline \mathrm{a} & 1 & 1 \\ \mathrm{b} & j & -j \end{array} \qquad 9.11$$
$$i^b = \tfrac{1}{2}j(i^1 - i^2)$$

There are no zero-sequence quantities.

The Four Networks Associated with Every Problem

(*a*) Let any three-phase network be given (Fig. 9.2*b*).

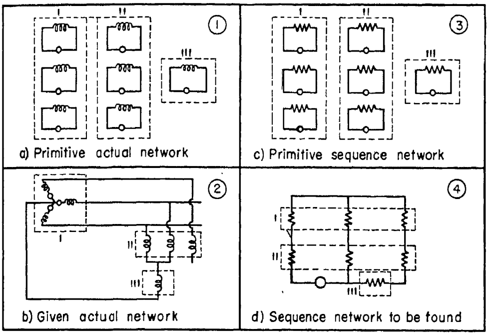


FIG. 9.2. The four basic reference frames of the method of symmetrical components.

When symmetrical components are to be used, four different networks and four different reference frames appear in the analysis in place of two (Fig. 9.2).

1. The primitive network of the given system having C coils and C meshes. It is always divided into several groups, each containing three coils (or one coil).
2. The given system with M meshes.

In both of these actual networks only actual currents flow.

3. The primitive network of the hypothetical sequence network also having C coils and C meshes in groups of three (or one) (the same number of groups as in the actual primitive system).

4. The hypothetical sequence network having the same number of meshes and coils as the given network, but a different number of sub-networks. This network, however, is unknown at the beginning of the analysis.

(*b*) In addition to the four *basic* networks, in which either all *actual* coils or all *sequence* coils appear, there are also two mixed networks containing both types of coils. That is, both the primitive and the inter-connected networks may contain both types of coils.

Considering the mixed primitive system, in the actual coils only actual currents flow; in the sequence coils, only sequence currents.

(*c*) There is now a large variety of ways in which the problem may be stated. For instance:

1. The self and mutual impedances (or impressed voltages) either in the primitive actual network 1 or in the sequence network 3 or partly in one, partly in the other, may be given.

2. The currents and voltages either of the given actual network 3 or of the sequence network 4 or both are wanted.

(*d*) It is possible to establish a **C** between any two of the four networks. In particular:

1. The **C** between the two *actual* networks 1 and 2 (that is, $\mathbf{C}_2^1$) is the usual **C** hitherto developed involving the constraints of Kirchhoff.

2. *Fortescue's* $\mathbf{C}_s$ *given in equation 9.7, the so-called sequence tensor, represents the transformation only between the two primitive networks 1 and 3, namely,* $\mathbf{C}_3^1$, *and even there it transforms only corresponding groups. For each group of three coils an additional* $\mathbf{C}_s$ *has to be used.*

3. *The main problem to be investigated presently is to find* **C** *changing network 3 to 4, if the usual* **C** *changing network 1 to 2 is known.*

Given Sequence Quantities, Find Actual Currents

In many special problems **Z** and **e** of network 3 are given and **e** and **i** of only network 2 are to be found. The steps are obvious.

1. Change **Z** and **e** from 3 to 1 by the sequence tensor $\mathbf{C}_s$.

2. From then on the analysis follows the same as usual.

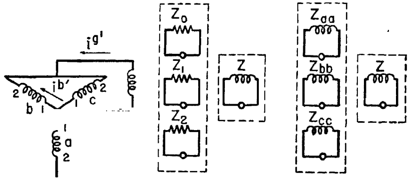
The only difference now is that the *components* of **Z** and **e** of 1 contain sequence impedances (and voltages) instead of the conventional actual impedances.

Quite often the design constants are given in both 1 and 3. That is, a "mixed" primitive system only is given. Usually the rotating machines are given along 3 and the stationary coils (the fault impedances) in 1. Under such conditions change **Z** of only the rotating machines by $\mathbf{C}_s$ from 3 to 1. From then on the analysis is as usual.

Example. For instance, let a generator be connected to a load as shown in Fig. 9.3*a*. The primitive system is known only as shown in Fig. 9.3*b*. That is:

1. The generator constants are given along the sequence axes

$$
\mathbf{Z}_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}
\begin{array}{c}
\begin{array}{ccc} 0 & 1 & 2 \end{array} \\
\left[\begin{array}{ccc} Z_0 & & \\ & Z_1 & \\ & & Z_2 \end{array}\right]
\end{array}
\qquad
\mathbf{e}_1 = \begin{array}{c}
\begin{array}{ccc} 0 & 1 & 2 \end{array} \\
\left[\begin{array}{ccc} & e_1 & \end{array}\right]
\end{array}
\qquad 9.12
$$



(*a*) Given system. (*b*) Mixed primitive. (*c*) Actual primitive.

FIG. 9.3. Generator connected to a load.

2. The network constant is given along the actual axes

$$
\mathbf{Z}_2 = \begin{array}{c} \\ \mathrm{g} \end{array}
\begin{array}{c} \mathrm{g} \\ \left[\begin{array}{c} Z \end{array}\right] \end{array}
\qquad
\mathbf{e}_2 = \mathrm{g}\left[\begin{array}{c} 0 \end{array}\right]
\qquad 9.13
$$

The first step is to change the sequence part of the primitive network of Fig. 9.3*b* to an actual primitive network, Fig. 9.3*c*, by $\mathbf{C}_s$, equation 9.7. Hence by $\mathbf{C}_t^* \cdot \mathbf{Z}_1 \cdot \mathbf{C}$ and $\mathbf{C}_t^* \cdot \mathbf{e}$

$$
\mathbf{Z}_1' = \tfrac{1}{3}
\begin{array}{c} \\ \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{array}
\begin{array}{c}
\begin{array}{ccc} \mathrm{a} & \mathrm{b} & \mathrm{c} \end{array} \\
\left[\begin{array}{ccc}
Z_0 + Z_1 + Z_2 & Z_0 + aZ_1 + a^2Z_2 & Z_0 + a^2Z_1 + aZ_2 \\
Z_0 + a^2Z_1 + aZ_2 & Z_0 + Z_1 + Z_2 & Z_0 + aZ_1 + a^2Z_2 \\
Z_0 + aZ_1 + a^2Z_2 & Z_0 + a^2Z_1 + aZ_1 & Z_0 + Z_1 + Z_2
\end{array}\right]
\end{array}
=
\begin{array}{c} \\ \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{array}
\begin{array}{c}
\begin{array}{ccc} \mathrm{a} & \mathrm{b} & \mathrm{c} \end{array} \\
\left[\begin{array}{ccc}
Z_{aa} & Z_{ab} & Z_{ac} \\
Z_{ba} & Z_{bb} & Z_{bc} \\
Z_{ca} & Z_{cb} & Z_{cc}
\end{array}\right]
\end{array}
\qquad 9.14
$$

$$
\mathbf{e}_1' = \frac{1}{\sqrt{3}}
\begin{array}{c}
\begin{array}{ccc} \mathrm{a} & \mathrm{b} & \mathrm{c} \end{array} \\
\left[\begin{array}{ccc} e_1 & ae_1 & a^2e_1 \end{array}\right]
\end{array}
$$

Now both groups of coils are reduced to the actual reference axes, and the analysis follows the usual steps. Assuming two independent currents in Fig. 9.3*a*,

$$i^a = 0$$
$$i^b = i^{b'}$$
$$i^c = -i^{b'} - i^{g'}$$
$$i^g = i^{g'}$$

$$\mathbf{C} = \begin{array}{c|cc} & b' & g' \\ \hline a & & \\ b & 1 & \\ c & -1 & -1 \\ g & & 1 \end{array} \qquad 9.15$$

By $\mathbf{C}_t^* \cdot (\mathbf{Z}_1 + \mathbf{Z}_2) \cdot \mathbf{C}$, and $\mathbf{C}_t^* \cdot (\mathbf{e}_1 + \mathbf{e}_2)$.

$$\mathbf{Z}' = \begin{array}{c|cc} & b' & g' \\ \hline b' & Z_1 + Z_2 & [Z_1(1-a) + Z_2(1-a^2)]/3 \\ g' & [Z_1(1-a^2) + Z_2(1-a)]/3 & (Z_0 + Z_1 + Z_2 + 3Z_g)/3 \end{array}$$

$$\mathbf{e}' = \frac{1}{\sqrt{3}} \begin{array}{|c|c|} \multicolumn{1}{c}{b'} & \multicolumn{1}{c}{g'} \\ \hline (a - a^2)e_1 & -a^2 e_1 \\ \hline \end{array} \qquad 9.16$$

Given Sequence Quantities, Find Sequence Currents

(*a*) Since in many problems **Z** of network 4 is simpler than **Z** of 2 (its inverse is easier to find), it is advantageous to find first the currents of 4, then change them to those of 2 (the actually existing currents). Also, since network 4 is simpler than 2 (containing more subnetworks), it is easier to use it on the a-c. network analyzer. Hence the establishment of network 4 is important.

(*b*) That is, the problem is as follows:

Given: networks 1, 2, and 3 (or their **Z** and **e**).
Find: network 4 (or its **Z**′ and **e**′).

Or stated in another way:

Given: $\mathbf{C}_2^1$ changing network 1 to 2.
Find: $\mathbf{C}_4^3$ changing network 3 to 4.

The difficulty of this step is that the *law of transformation for* **C** (equation 6.11) *cannot be used* since here only one of the needed **C**'s is

available (from 1 to 3, namely, the sequence tensor). The other **C** (from 2 to 4) is not yet available, as network 4 is unknown.

Steps in Changing C_2^1 to C_4^3

(*a*) Even though the law of transformation of C_2^1 cannot be used, still C_4^3 can be found by the following steps:

1. Let C_2^1 be rearranged as a compound tensor

$$C_2^1 = \boxed{\begin{array}{c} I \\ \hline C' \end{array}} \qquad 9.17$$

2. Let Fortescue's transformation C_s (containing as many of equation 9.7 as there are groups of three coils) also be expressed as a compound tensor

$$C_s = \boxed{\begin{array}{c|c} C_1 & C_2 \\ \hline C_3 & C_4 \end{array}} \qquad 9.18$$

The sequence axes (written on top of C_s) have to be arranged in the same order as the real axes in C_2^1. That is, first the independent (the "new") sequence axes are written, then the rest that are to be eliminated.

3. *The desired transformation tensor of the sequence network is found by* (proof to follow)

$$C_4^3 = \boxed{\begin{array}{c} I \\ \hline -(C'\cdot C_2 - C_4)^{-1}\cdot(C'\cdot C_1 - C_3) \end{array}} \qquad 9.19$$

It is necessary to calculate the inverse of a matrix having as many rows as there are equations of constraint.

(*b*) Since the inverse calculation of C_s is simple, the above equation may also be written as

$$C_4^3 = \boxed{\begin{array}{c} I \\ \hline -(C_4^{-1}\cdot C'\cdot C_2 - I)^{-1}\cdot C_4^{-1}\cdot(C'\cdot C_1 - C_3) \end{array}} \qquad 9.20$$

In many cases the use of this formula requires fewer calculations.

(c) Once **C** from network 3 to 4 has been found, then **Z** and **e** of network 4 are found from those of 3 by the usual formulas. Also *the sequence network* 4 (containing sequence coils) may be established from $\mathbf{C}_4^3$ by inspection.

The sequence network 4 containing the real coils may be established (using **Z** and **e** of network 1) by finding $\mathbf{C}_4^1$. Once $\mathbf{C}_4^3$ is known, then the former is found by

$$\mathbf{C}_4^1 = \mathbf{C}_3^1 \cdot \mathbf{C}_4^3 \tag{9.21}$$

where $\mathbf{C}_3^1$ is the Fortescue's tensor $\mathbf{C}_s$ shown in equation 9.18.

(d) When the hypothetical $\mathbf{i}^4$ have been found, then the actual currents $\mathbf{i}^1$ flowing in each actual coil are found by

$$\mathbf{i}^1 = \mathbf{C}_4^1 \cdot \mathbf{i}^4 \tag{9.22}$$

Proof of the Formula Changing $\mathbf{C}_2^1$ to $\mathbf{C}_4^3$

It was shown in equation 7.13 that, if the unit tensor **i** is subtracted from $\mathbf{C}_2^1$, the resultant **B** multiplied by **i** gives the equations of constraint $\mathbf{B} \cdot \mathbf{i} = 0$, where

$$\mathbf{B} = \begin{array}{|c|c|} \hline & \\ \hline \mathbf{C}' & -\mathbf{I} \\ \hline \end{array}$$

Now if **i** is replaced by $\mathbf{C}_s \cdot \mathbf{i}'$, the equations of constraint $\mathbf{B}' \cdot \mathbf{i}' = 0$ of the sequence network are found, where

$$\mathbf{B}' = \mathbf{B} \cdot \mathbf{C}_s = \begin{array}{|c|c|} \hline & \\ \hline \mathbf{C}' & -\mathbf{I} \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline \mathbf{C}_1 & \mathbf{C}_2 \\ \hline \mathbf{C}_3 & \mathbf{C}_4 \\ \hline \end{array} = \begin{array}{|c|c|} \hline & \\ \hline \mathbf{C}' \cdot \mathbf{C}_1 - \mathbf{C}_3 & \mathbf{C}' \cdot \mathbf{C}_2 - \mathbf{C}_4 \\ \hline \end{array} = \begin{array}{|c|c|} \hline & \\ \hline \mathbf{B}_a & \mathbf{B}_b \\ \hline \end{array}$$

or

$$\mathbf{B}' \cdot \mathbf{i}' = \mathbf{B}_a \mathbf{i}^a + \mathbf{B}_b \cdot \mathbf{i}^b$$

Expressing now the dependent currents in terms of the independents

$$\mathbf{i}^b = -\mathbf{B}_b^{-1} \cdot \mathbf{B}_a \cdot \mathbf{i}^a = -(\mathbf{C}' \cdot \mathbf{C}_2 - \mathbf{C}_4)^{-1} \cdot (\mathbf{C}' \cdot \mathbf{C}_1 - \mathbf{C}_3) \cdot \mathbf{i}^a \tag{9.23}$$

giving the lower part of $\mathbf{C}_4^3$.

Changing Reference Frames of Faults

(a) In fault studies there are a few standardized types of impedances whose **Z** has to be changed from network 1 to 3 with the aid of $\mathbf{C}_s$.

To save the repetition of transformation, Table III lists the **Z** of frequent impedance combinations for both reference frames. All are special cases of the first set, by making some of the **Z**'s zero. (Similar tables are shown in *T.A.N.*, Chapters XIX–XX, for other standard three-phase networks.)

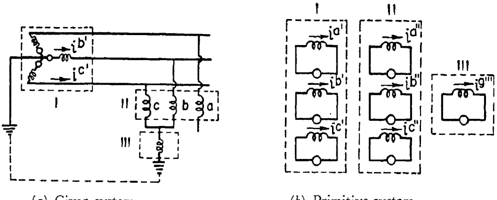
(*b*) A ground coil may have special treatment. To avoid the use of three-rowed matrices, a ground coil is considered to have only a zero-sequence impedance as shown at the end of Table III.

Also an actual ground current i^g is transformed into a zero-sequence current by the following transformation:

$$i^g = \sqrt{3}\, i^0 \quad \text{or} \quad \mathrm{C} = \begin{array}{c|c|} & 0 \\ \hline g & \sqrt{3} \\ \hline \end{array} = \frac{1}{\sqrt{3}} \begin{array}{c|c|} & 0 \\ \hline g & 3 \\ \hline \end{array} \qquad 9.24$$

The reason is that the ground carries the sum of the three currents

$$\begin{aligned} i^g &= i^a + i^b + i^c \\ &= \frac{1}{\sqrt{3}}\left[(i^0 + i^1 + i^2) + (i^0 + a^2 i^1 + a i^2) + (i^0 + a i^1 + a^2 i^2)\right] \\ &= 3i^0/\sqrt{3} = \sqrt{3}i^0 \end{aligned}$$



(*a*) Given system. (*b*) Primitive system.

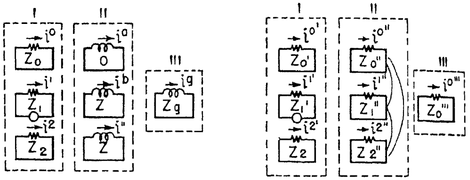
FIG. 9.4. Double line-to-ground short circuit.

Changing A "Mixed" Primitive to a Primitive Sequence Network

Let the network of Fig. 9.4*a* (a double line-to-ground short circuit) be analyzed whose design constants are given in the form of the mixed primitive of Fig. 9.5*a*.

$$\mathbf{Z}_1 = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & Z_0 & & \\ 1 & & Z_1 & \\ 2 & & & Z_2 \\ \hline \end{array} \qquad \mathbf{Z}_2 = \begin{array}{c|ccc|} & a & b & c \\ \hline a & & & \\ b & & Z & \\ c & & & Z \\ \hline \end{array} \qquad 9.25$$

$$\mathbf{e}_1 = \begin{array}{|ccc|} 0 & 1 & 2 \\ \hline & e_1 & \\ \hline \end{array} \qquad \mathbf{Z}_3 = \begin{array}{c|c|} & g \\ \hline g & Z_g \\ \hline \end{array}$$



(a) Given impedances of mixed primitive network.

(b) Changed to impedances of sequence primitive 3.

FIG. 9.5. Known impedances.

The first step is to change the mixed primitive into the sequence primitive by changing $\mathbf{Z}_1$ with the aid of Table III. Hence $\mathbf{Z}$ and $\mathbf{e}$ of the primitive sequence (Fig. 9.5b) are

$$\mathbf{Z} = \frac{1}{3}\begin{array}{c|ccc|ccc|c|} & 0' & 1' & 2' & 0'' & 1'' & 2'' & 0''' \\ \hline 0' & 3Z_0 & & & & & & \\ 1' & & 3Z_1 & & & & & \\ 2' & & & 3Z_2 & & & & \\ \hline 0'' & & & & 2Z & -Z & -Z & \\ 1'' & & & & -Z & 2Z & -Z & \\ 2'' & & & & -Z & -Z & 2Z & \\ \hline 0''' & & & & & & & Z_g \\ \hline \end{array} \qquad \mathbf{e} = \begin{array}{c|c|} 0' & \\ 1' & e_1 \\ 2' & \\ \hline 0'' & \\ 1'' & \\ 2'' & \\ \hline 0''' & \\ \hline \end{array} \qquad 9.26$$

Note that ⅓ is factored out.

Changing C_2^1 into C_4^3

C of Fig. 9.4, changing network 1 to 2 (showing the manner of interconnection of coils), is

$$
\begin{aligned}
i^{a'} &= 0 \\
i^{b'} &= i^{b'} \\
i^{c'} &= i^{c'} \\
i^{a''} &= 0 \\
i^{b''} &= i^{b'} \\
i^{c''} &= i^{c'} \\
i^{g'''} &= i^{b'} + i^{c'}
\end{aligned}
\qquad
C_2^1 = \begin{array}{c|c|c|}
 & b' & c' \\
\hline
a' & & \\
b' & 1 & \\
c' & & 1 \\
a'' & & \\
b'' & 1 & \\
c'' & & 1 \\
g''' & 1 & 1 \\
\hline
\end{array}
\tag{9.27}
$$

Two of the equations (second and third) are the equations of independence; hence the remaining five are the equations of constraint.

1. Rearranging

$$
C_2^1 = \begin{array}{|c|}
\hline
I \\
\hline
C' \\
\hline
\end{array}
= \begin{array}{c|c|c|}
 & b' & c' \\
\hline
b' & 1 & \\
c' & & 1 \\
\hline
d' & & \\
d'' & & \\
b'' & 1 & \\
c'' & & 1 \\
g''' & 1 & 1 \\
\hline
\end{array}
\tag{9.28}
$$

2. Fortescue's transformation is, by equations 9.7 and 9.24,

$$
C_3^1 = C_s = \begin{array}{|c|c|}
\hline
C_1 & C_2 \\
\hline
C_3 & C_4 \\
\hline
\end{array}
= \frac{1}{\sqrt{3}}
\begin{array}{c|c|c|c|c|c|c|c|}
 & 1' & 2' & 0' & 0'' & 1'' & 2'' & 0''' \\
\hline
b' & a^2 & a & 1 & & & & \\
c' & a & a^2 & 1 & & & & \\
\hline
d' & 1 & 1 & 1 & & & & \\
d'' & & & & 1 & 1 & 1 & \\
b'' & & & & 1 & a^2 & a & \\
c'' & & & & 1 & a & a^2 & \\
g''' & & & & & & & 3 \\
\hline
\end{array}
\tag{9.29}
$$

3. Performing the indicated multiplication,

$$\mathrm{C'}\cdot\mathrm{C}_2 - \mathrm{C}_4 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 & & & & \\ & -1 & -1 & -1 & \\ 1 & -1 & -a^2 & -a & \\ 1 & -1 & -a & -a^2 & \\ 2 & & & & -3 \end{bmatrix}$$

Its inverse is

$$(\mathrm{C'}\cdot\mathrm{C}_2 - \mathrm{C}_4)^{-1} = \frac{1}{\sqrt{3}}\begin{bmatrix} -3 & & & & \\ -2 & -1 & -1 & -1 & \\ 1 & -1 & -a & -a^2 & \\ 1 & -1 & -a^2 & -a & \\ -2 & & & & -1 \end{bmatrix} \qquad \mathrm{C'}\cdot\mathrm{C}_1 - \mathrm{C}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 & -1 \\ & \\ a^2 & a \\ a & a^2 \\ -1 & -1 \end{bmatrix}$$

The product of the last two matrices is

$$(\mathrm{C'}\cdot\mathrm{C}_2 - \mathrm{C}_4)^{-1}\cdot(\mathrm{C'}\cdot\mathrm{C}_1 - \mathrm{C}_3) = \begin{array}{c} \\ 0' \\ 0'' \\ 1'' \\ 2'' \\ 0''' \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & \\ & -1 \\ 1 & 1 \end{bmatrix} \end{array} \qquad 9.30$$

Hence the desired transformation tensor is

$$\mathrm{C}_4^3 = \begin{bmatrix} \mathrm{I} \\ \mathrm{C'} \end{bmatrix} = \begin{array}{c} \\ 1' \\ 2' \\ 0' \\ 0'' \\ 1'' \\ 2'' \\ 0''' \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{bmatrix} 1 & \\ & 1 \\ -1 & -1 \\ -1 & -1 \\ 1 & \\ & 1 \\ -1 & -1 \end{bmatrix} \end{array} \qquad 9.31$$

The Equations of the Sequence Network

(*a*) Now **Z** and **e** of the primitive network, equation 9.26, may be transformed by $\mathbf{C}_4^3$ with the aid of $\mathbf{C}_t^* \cdot \mathbf{Z} \cdot \mathbf{C}$ and $\mathbf{C}_t^* \cdot \mathbf{e}$.

$$\mathbf{Z}\cdot\mathbf{C} = \begin{array}{c|c|c|} & 1' & 2' \\ \hline 0' & -Z_0 & -Z_0 \\ 1' & Z_1 & \\ 2' & & Z_2 \\ \hline 0'' & -Z & -Z \\ 1'' & Z & \\ 2'' & & Z \\ \hline 0''' & -3Z_g & -3Z_g \\ \hline \end{array} \qquad 9.32$$

$$\mathbf{C}_t^* \cdot \mathbf{Z} \cdot \mathbf{C} = \mathbf{Z}' = \begin{array}{c|c|c|} & 1' & 2' \\ \hline 1' & Z_0 + Z_1 + 2Z + 3Z_g & Z_0 + Z + 3Z_g \\ \hline 2' & Z_0 + Z + 3Z_g & Z_0 + Z_2 + 2Z + 3Z_g \\ \hline \end{array} \qquad 9.33$$

$$\mathbf{C}_t^* \cdot \mathbf{e} = \mathbf{e}' = \begin{array}{|c|c|} 1' & 2' \\ \hline e_1 & \\ \hline \end{array}$$

The equations of the sequence network are $\mathbf{e}' = \mathbf{Z}' \cdot \mathbf{i}'$ or $\mathbf{e}_4 = \mathbf{Z}_{44} \cdot \mathbf{i}^4$

$$\begin{aligned} e_1 &= (Z_0 + Z_1 + 2Z + 3Z_g)i^{1'} + (Z_0 + Z + 3Z_g)i^{2'} \\ 0 &= (Z_0 + Z + 3Z_g)i^{1'} + (Z_0 + Z_2 + 2Z + 3Z_g)i^{2'} \end{aligned} \qquad 9.34$$

They may be solved for the currents $i^{1'}$ and $i^{2'}$ by $\mathbf{i}' = \mathbf{Z}'^{-1} \cdot \mathbf{e}'$.

(*b*) If the sequence currents $i^{1'}$ and $i^{2'}$ have been found, then in network 3:

1. By equation 9.31, the sequence currents are $\mathbf{i}^3 = \mathbf{C}_4^3 \cdot \mathbf{i}^4$

$$\mathbf{i}^4 = \begin{array}{|c|c|c|c|c|c|c|} 0' & 1' & 2' & 0'' & 1'' & 2'' & 0''' \\ \hline -i^{1'} - i^{2'} & i^{1'} & i^{2'} & -i^{1'} - i^{2'} & i^{1'} & i^{2'} & -i^{1'} - i^{2'} \\ \hline \end{array} \qquad 9.35$$

2. The sequence voltages are, by $\mathbf{Z}\cdot\mathbf{C}\cdot\mathbf{i}'$, where $\mathbf{Z}\cdot\mathbf{C}$ is given in equation 9.32,

$$\mathbf{e} = \begin{array}{|c|c|c|c|c|c|c|} 0' & 1' & 2' & 0'' & 1'' & 2'' & 0''' \\ \hline -Z_0(i^{1'} + i^{2'}) & Z_1 i^{1'} & Z_2 i^{2'} & -Z(i^{1'} + i^{2'}) & Z i^{1'} & Z i^{2'} & -3Z_g(i^{1'} + i^{2'}) \\ \hline \end{array} \qquad 9.36$$

Actual Currents and Differences of Potential

A transformation from network 3 to network 2 with the aid of $\mathbf{C}_3^1 = \mathbf{C}_s$ shown in equation 9.29 gives the *actual* currents in each coil as

$$\mathbf{i}^1 = \mathbf{C}_3^1 \cdot \mathbf{i}^3 = \frac{1}{\sqrt{3}} \begin{array}{c|c|} \mathrm{a}' & 0 \\ \mathrm{b}' & (a^2-1)i^{1'} + (a-1)i^{2'} \\ \mathrm{c}' & (a-1)i^{1'} + (a^2-1)i^{2'} \\ \mathrm{a}'' & 0 \\ \mathrm{b}'' & (a^2-1)i^{1'} + (a-1)i^{2'} \\ \mathrm{c}'' & (a-1)i^{1'} + (a^2-1)i^{2'} \\ \mathrm{g}''' & -3(i^{1'} + i^{2'}) \end{array} \qquad 9.37$$

The *actual* difference of potential across each coil (since $\mathbf{e}' = \mathbf{C}_t^* \cdot \mathbf{e}$ or $\mathbf{e}_3 = \mathbf{C}_{3t}^{1*} \cdot \mathbf{e}_1$, therefore $\mathbf{e}_1 = (\mathbf{C}_{3t}^{1*})^{-1} \cdot \mathbf{e}_3$) is

$$\mathbf{e}_1 = (\mathbf{C}_{3t}^{1*})^{-1} \cdot \mathbf{e}_3 = \frac{1}{\sqrt{3}} \begin{array}{c|c|} \mathrm{a}' & -Z_0(i^{1'} + i^{2'}) + Z_1 i^{1'} + Z_2 i^{2'} \\ \mathrm{b}' & -Z_0(i^{1'} + i^{2'}) + a^2 Z_1 i^{1'} + a Z_2 i^{2'} \\ \mathrm{c}' & -Z_0(i^{1'} + i^{2'}) + a Z_1 i^{1'} + a^2 Z_2 i^{2'} \\ \mathrm{a}'' & 0 \\ \mathrm{b}'' & Z(a^2-1)i^{1'} + (a-1)i^{2'} \\ \mathrm{c}'' & Z(a-1)i^{1'} + (a^2-1)i^{2'} \\ \mathrm{g}''' & -3Z_g(i^{1'} + i^{2'}) \end{array} \qquad 9.38$$

Of course the *actual* currents $\mathbf{i}^1$ and differences of potentials $\mathbf{e}_1$ could have been found from $\mathbf{i}^4$ without the intermediary steps of finding $\mathbf{i}^3$ and $\mathbf{e}_3$. That is

$$\mathbf{i}^1 = \mathbf{C}_4^1 \cdot \mathbf{i}^4 = \mathbf{C}_3^1 \cdot \mathbf{C}_4^3 \cdot \mathbf{i}^4 \qquad 9.39$$

$$\mathbf{e}_1 = (\mathbf{C}_{3t}^{1*})^{-1} \cdot \mathbf{e}_3 = (\mathbf{C}_{3t}^{1*})^{-1} \cdot \mathbf{Z} \cdot \mathbf{C}_4^3 \cdot \mathbf{i}^4 \qquad 9.40$$

The Sequence Network

When $\mathbf{Z}'$, equation 9.33, has been established, the sequence network containing the design constants of the *mixed* primitive network of Fig. 9.4*b* may be established by inspection as shown in Fig. 9.6.

From $\mathbf{C}_4^3$, equation 9.31, the sequence network containing only sequence constants is established by inspection, as shown in Fig. 9.7.

There are mutual inductances between the double-primed coils as shown in Fig. 9.5.

It should be noted that Fig. 9.6 is a "mixed" network containing both actual and sequence impedances and that no mutual inductances exist between the coils. This mixed circuit can be set up on the calculating board. However, in both sequence networks, Figs. 9.6 and 9.7, the currents are all sequence currents.

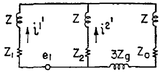


FIG. 9.6. Sequence network with mixed impedances.

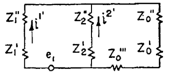


FIG. 9.7. Sequence network with sequence impedances.

Networks with Multiwinding Transformers

When magnetizing currents are also to be neglected, the steps are the same as before except that $\mathbf{C}_2^1$ is the product of two $\mathbf{C}$'s.

Because of the larger number of groups of coils, in such problems it is advantageous to deal with compound networks (each coil representing a three-phase apparatus) and their compound tensors. It is shown elsewhere * that *in terms of three-phase compound tensors the analysis of three-phase networks reduces almost to the simplicity of that of single-phase networks.*

The advantage of the use of compound tensors is due to the fact that only a few standardized types of three-phase interconnections and faults exist and their corresponding $\mathbf{C}$ and $\mathbf{Z}$ need be established only once. Then for every particular three-phase system these *ready-made* $\mathbf{C}$'s and $\mathbf{Z}$'s are used as components of the compound tensors.

EXERCISES

1. Find the conjugate of

$$\mathbf{e} = \begin{array}{|c|c|}\hline e_1 & e_2 - je_3 \\ \hline\end{array}$$

$$\mathbf{e} = \begin{array}{|c|c|}\hline \epsilon^{j\theta} & \\ \hline & \epsilon^{-j\theta} \\ \hline\end{array} \qquad \mathbf{Z} = \begin{array}{|c|c|}\hline p - jp\theta & Z\epsilon^{j\alpha} \\ \hline Z & p + jp\theta \\ \hline\end{array}$$

* *T.A.N.*, Chapter XX.

2(*a*) Transform to 0, 1, 2 axes the following Z's.

$$\mathbf{Z}_1 = \begin{array}{c|ccc|} & a & b & c \\ \hline a & Z_a & & \\ b & & Z_b & \\ c & & & Z_c \\ \hline \end{array} \qquad \mathbf{Z}_2 = \begin{array}{c|ccc|} & a & b & c \\ \hline a & Z_{aa} & Z_{ab} & Z_{ac} \\ b & Z_{ba} & Z_{bb} & Z_{bc} \\ c & Z_{ca} & Z_{cb} & Z_{cc} \\ \hline \end{array}$$

(*b*) Transform the new Z's back to their original form.

3. Verify the fault impedances **Z** in Table III.

4. Let the generator of Fig. 9.8, whose **Z** and **e** are given in equation 9.12 supplying three unequal resistences, be short-circuited as shown. Find the short-circuit current.

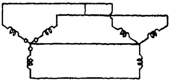


Fig. 9.8.

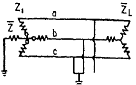


Fig. 9.9.

5. A grounded generator supplies a three-phase load where

$$\mathbf{Z}_1 = \begin{array}{c|ccc|} & 0' & 1' & 2' \\ \hline 0' & Z_0 & & \\ 1' & & Z_1 & \\ 2' & & & Z_2 \\ \hline \end{array} \qquad \mathbf{Z}_L = \begin{array}{c|ccc|} & 0'' & 1'' & 2'' \\ \hline 0'' & & & \\ 1'' & & Z_{L1} & \\ 2'' & & & Z_{L2} \\ \hline \end{array} \qquad \mathbf{Z} = \begin{array}{c|c|} & 0''' \\ \hline 0''' & 3Z_g \\ \hline \end{array}$$

$$\mathbf{e} = \begin{array}{|ccc|} 0' & 1' & 2' \\ \hline & e_1 & \\ \hline \end{array}$$

If a double line-to-ground short circuit occurs as in Fig. 9.9, find the sequence and the *actual* currents and differences of potentials appearing across each phase of the generator and the load.

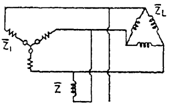


Fig. 9.10.

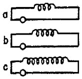
6. A generator $\mathbf{Z}_1$ supplies a delta-connected load $\mathbf{Z}_L$. If a line-to-line fault occurs through an impedance **Z** (Fig. 9.10), what are the sequence and the actual currents and voltages in each coil?

$$\mathbf{Z}_1 = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & Z_0 & & \\ 1 & & Z_1 & \\ 2 & & & Z_2 \\ \hline \end{array} \qquad \mathbf{Z}_L = \begin{array}{c|ccc|} & a & b & c \\ \hline a & Z & & \\ b & & Z & \\ c & & & Z \\ \hline \end{array}$$

TABLE III

Fault Impedances along Actual and along Sequence Axes

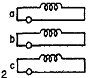
1



$$Z = \begin{array}{c|ccc|} & a & b & c \\ \hline a & Z_a & & \\ b & & Z_b & \\ c & & & Z_c \\ \hline \end{array} = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & Z_0 & Z_2 & Z_1 \\ 1 & Z_1 & Z_0 & Z_2 \\ 2 & Z_2 & Z_1 & Z_0 \\ \hline \end{array}$$

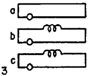
$$Z_0 = (Z_a + Z_b + Z_c)/3$$
$$Z_1 = (Z_o + aZ_b + a^2 Z_c)/3$$
$$Z_2 = (Z + a^2 Z_b + aZ_c)/3$$

2



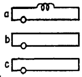
$$Z = \begin{array}{c|ccc|} & a & b & c \\ \hline a & Z & & \\ b & & Z & \\ c & & & Z \\ \hline \end{array} = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & Z & & \\ 1 & & Z & \\ 2 & & & Z \\ \hline \end{array}$$

3



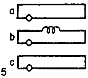
$$Z = \begin{array}{c|ccc|} & a & b & c \\ \hline a & & & \\ b & & Z & \\ c & & & Z \\ \hline \end{array} = \frac{1}{3} \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & 2Z & -Z & -Z \\ 1 & -Z & 2Z & -Z \\ 2 & -Z & -Z & 2Z \\ \hline \end{array}$$

4



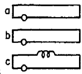
$$Z = \begin{array}{c|ccc|} & a & b & c \\ \hline a & Z & & \\ b & & & \\ c & & & \\ \hline \end{array} = \frac{1}{3} \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & Z & Z & Z \\ 1 & Z & Z & Z \\ 2 & Z & Z & Z \\ \hline \end{array}$$

5



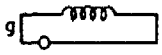
$$Z = \begin{array}{c|ccc|} & a & b & c \\ \hline a & & & \\ b & & Z & \\ c & & & \\ \hline \end{array} = \frac{1}{3} \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & Z & a^2Z & a\,Z \\ 1 & a\,Z & Z & a^2Z \\ 2 & a^2Z & a\,Z & Z \\ \hline \end{array}$$

6



$$Z = \begin{array}{c|ccc|} & a & b & c \\ \hline a & & & \\ b & & & \\ c & & & Z \\ \hline \end{array} = \frac{1}{3} \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & Z & a\,Z & a^2Z \\ 1 & a^2Z & Z & a\,Z \\ 2 & a\,Z & a^2Z & Z \\ \hline \end{array}$$

7



$$Z = \begin{array}{c|c|} & g \\ \hline g & Z_g \\ \hline \end{array} = \begin{array}{c|c|} & 0 \\ \hline 0 & 3Z_g \\ \hline \end{array}$$

CHAPTER 10

MERCURY-ARC RECTIFIER CIRCUITS

Information Implied in C

The connection tensor **C** showing how the coils are connected into a network includes a surprising amount of information about the network. It will be shown that in rectifier circuits it gives the instantaneous and r.m.s. values of the currents flowing at any part of the system, provided that the load current is not too large. $\mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C}$, of course, gives the impedance of the network to be used in detailed studies.

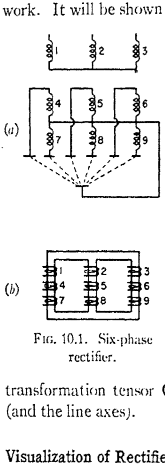


FIG. 10.1. Six-phase rectifier.

As a simple example, let the six-phase rectifier of Fig. 10.1*a* be considered. Each anode circuit is considered a closed mesh. The nine coils are wound on a transformer as shown in Fig. 10.1*b*.

The method of attack is the same as that of any other network containing multiwinding transformers. That is:

1. The coils are interconnected by $\mathbf{C}_1$.
2. The magnetizing currents are neglected (or retained) by $\mathbf{C}_2$.
3. The product $\mathbf{C}_1 \cdot \mathbf{C}_2$ gives the desired transformation tensor $\mathbf{C}'$ in which the new axes are the anode axes (and the line axes).

Visualization of Rectifier Phenomena

(*a*) Hitherto no attention was paid to the order of the meshes assumed. Now, however, it is essential to rearrange the anode meshes *in the order of their firing*.

The order of firing of the various anodes is determined by the equation $\mathbf{e}' = \mathbf{C}_t \cdot \mathbf{e}$, where **e** is the impressed line voltage (three-phase). The components of $\mathbf{e}'$ give the differences of potentials appearing across the various anodes. By arranging the components of $\mathbf{e}'$ in their proper time phase, the firing order of the anodes is automatically determined.

A transformation tensor $\mathbf{C}_f$ may now be established that changes the order of the anodes to that given by $\mathbf{e}'$. Then $\mathbf{C}' \cdot \mathbf{C}_f$ is the final $\mathbf{C}$ sought.

(*b*) Now if it is assumed that the new currents $\mathbf{i}'$ (the anode currents) that appear at different time intervals are all equal and of constant value $I_{\text{d-c.}}$, then $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$ gives the currents flowing in each coil in each time interval in terms of the direct current $I_{\text{d-c.}}$.

Since the anode currents $I_{\text{d-c.}}$ appear at different time intervals in each anode, *the graphical plot of each row of* $\mathbf{C}$ (multiplied by $I_{\text{d-c.}}$) *gives the instantaneous value of the currents flowing in each coil.*

Also if the components of each row of $\mathbf{C}$ are squared, then added and the square root of the average resultant is taken, the r.m.s. value of the currents flowing in the corresponding coils is found.

Six-Phase Rectifier

(*a*) In the arrangement of Fig. 10.1 three of the coils (the primary) are connected in star to the line. Each of the other six coils (the secondary) forms a closed mesh through the cathode. Coils 7–9 are connected to the common cathode in a direction opposite to coils 4–6. Hence their interconnection is represented by

$$\mathbf{C}_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{cccccccc} 1' & 2' & 4' & 5' & 6' & 7' & 8' & 9' \\ \hline 1 & & & & & & & \\ & 1 & & & & & & \\ -1 & -1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & -1 & & \\ & & & & & & -1 & \\ & & & & & & & -1 \end{array} \qquad 10.1$$

(*b*) There are two closed magnetic meshes; hence two equations of constraint may be set up:

$$\begin{aligned} n_p i^1 + n_s i^4 + n_s i^7 - n_p i^2 - n_s i^5 - n_s i^8 &= 0 \\ n_p i^2 + n_s i^5 + n_s i^8 - n_p i^3 - n_s i^6 - n_s i^9 &= 0 \end{aligned} \qquad 10.2$$

(The magnetizing currents on the right-hand side, instead of being equated to zero, may be assumed to be i^m and i^n.)

The primed currents being substituted, then $i^{1'}$ and $i^{2'}$ eliminated,

$$
\begin{aligned}
i^{1'} &= \frac{1}{3}\frac{n_s}{n_p}(-2i^{4'} + i^{5'} + i^{6'} + 2i^{7'} - i^{8'} - i^{9'}) \\
i^{2'} &= \frac{1}{3}\frac{n_s}{n_p}(i^{4'} - 2i^{5'} + i^{6'} - i^{7'} + 2i^{8'} - i^{9'})
\end{aligned}
\qquad 10.3
$$

If $n_s/n_p = n$,

$$
C_2 = \begin{array}{c} \\ 1' \\ 2' \\ 4' \\ 5' \\ 6' \\ 7' \\ 8' \\ 9' \end{array}
\begin{array}{c}
\begin{array}{cccccc} 4'' & 5'' & 6'' & 7'' & 8'' & 9'' \end{array} \\
\left[\begin{array}{ccc|ccc}
-\frac{2}{3}n & \frac{1}{3}n & \frac{1}{3}n & \frac{2}{3}n & -\frac{1}{3}n & -\frac{1}{3}n \\
\frac{1}{3}n & -\frac{2}{3}n & \frac{1}{3}n & -\frac{1}{3}n & \frac{2}{3}n & -\frac{1}{3}n \\
\hline
1 & & & & & \\
 & 1 & & & & \\
 & & 1 & & & \\
\hline
 & & & 1 & & \\
 & & & & 1 & \\
 & & & & & 1
\end{array}\right]
\end{array}
\qquad 10.4
$$

$$
C_1 \cdot C_2 = C' = n \times \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 4'' & 5'' & 6'' & 7'' & 8'' & 9'' \end{array} \\
\left[\begin{array}{ccc|ccc}
-2/3 & 1/3 & 1/3 & 2/3 & -1/3 & -1/3 \\
1/3 & -2/3 & 1/3 & -1/3 & 2/3 & -1/3 \\
1/3 & 1/3 & -2/3 & -1/3 & -1/3 & 2/3 \\
\hline
1/n & & & & & \\
 & 1/n & & & & \\
 & & 1/n & & & \\
\hline
 & & & -1/n & & \\
 & & & & -1/n & \\
 & & & & & -1/n
\end{array}\right]
\end{array}
\qquad 10.5
$$

(c) If the impressed voltage vector is

$$
e = \begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\left[\begin{array}{ccc|ccc|ccc}
e & ae & a^2e & & & & & &
\end{array}\right]
\end{array}
$$

then

$$e' = C'_t \cdot e = e \times \begin{array}{cccccc} 4'' & 5'' & 6'' & 7'' & 8'' & 9'' \\ \hline \multicolumn{1}{|c|}{-1} & \multicolumn{1}{c|}{-a} & \multicolumn{1}{c|}{-a^2} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{a} & \multicolumn{1}{c|}{a^2} \\ \hline \end{array} \qquad 10.6$$

From Fig. 10.2 the firing order $a, b, \cdots, f$ follows as 7'', 6'', 8'', 4'', 9'', 5'' (the order may start at any coil).

(*d*) The order of the coils in C' may be changed by the following transformation:*

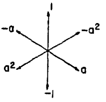


FIG. 10.2. Firing order.

$$\begin{aligned} i^{4''} &= i^d \\ i^{5''} &= i^f \\ i^{6''} &= i^b \\ i^{7''} &= i^a \\ i^{8''} &= i^c \\ i^{9''} &= i^e \end{aligned} \qquad C_3 = \begin{array}{c|cccccc|} & a & b & c & d & e & f \\ \hline 4'' & & & & 1 & & \\ 5'' & & & & & & 1 \\ 6'' & & 1 & & & & \\ 7'' & 1 & & & & & \\ 8'' & & & 1 & & & \\ 9'' & & & & & 1 & \\ \hline \end{array} \qquad 10.7$$

Hence

$$C = C' \cdot C_3 = n \times \begin{array}{c|cccccc|} & a & b & c & d & e & f \\ \hline 1 & 2/3 & 1/3 & -1/3 & -2/3 & -1/3 & 1/3 \\ 2 & -1/3 & 1/3 & 2/3 & 1/3 & -1/3 & -2/3 \\ 3 & -1/3 & -2/3 & -1/3 & 1/3 & 2/3 & 1/3 \\ 4 & & & & 1/n & & \\ 5 & & & & & & 1/n \\ 6 & & 1/n & & & & \\ 7 & -1/n & & & & & \\ 8 & & & -1/n & & & \\ 9 & & & & & -1/n & \\ \hline \end{array} \qquad 10.8$$

* *T.A.N.*, pp. 164–167.

(e) The instantaneous values of the currents are given by the rows of C multiplied by $I_{\text{d-c.}}$ as shown in Fig. 10.3 for the three line coils 1, 2, and 3.

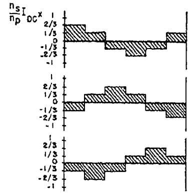


FIG. 10.3. Instantaneous line currents.

The r.m.s. current in coil 1 is

$$\mathbf{i} = I_{\text{d-c.}} \sqrt{\frac{(\frac{2}{3})^2 + (\frac{1}{3})^2 + (-\frac{1}{3})^2 + (-\frac{2}{3})^2 + (-\frac{1}{3})^2 + (\frac{1}{3})^2}{6}}$$

$$= \frac{\sqrt{2}}{3} I_{\text{d-c.}} \qquad 10.9$$

(f) The impedance tensor $\mathbf{Z}'$ of the system is $\mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C}$. It is needed in load and short-circuit studies.

Interphase Reactors

When the phases are interconnected through reactors (Fig. 10.4) then the anode does not stop firing as the next one starts. It keeps on firing through two or more time intervals.

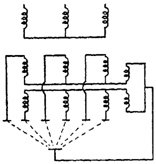


FIG. 10.4. Interphase reactors.

In such cases an additional $\mathbf{C}_4$ is introduced to show that in a time interval two (or more) consecutive anodes are firing simultaneously. For the above example

$$C_4 = \frac{1}{2}\;\begin{array}{c|ccc|ccc|} & a' & b' & c' & d' & e' & f' \\ \hline a & 1 & & & & & 1 \\ b & 1 & 1 & & & & \\ c & & 1 & 1 & & & \\ \hline d & & & 1 & 1 & & \\ e & & & & 1 & 1 & \\ f & & & & & 1 & 1 \\ \hline \end{array} \qquad 10.10$$

so that $C = C_1 \cdot C_2 \cdot C_3 \cdot C_4 =$

$$C = \frac{n}{2} \times \begin{array}{c|ccc|ccc|} & a' & b' & c' & d' & e' & f' \\ \hline 1 & 1 & 0 & -1 & -1 & 0 & 1 \\ 2 & 0 & 1 & 1 & -1 & -1 & 0 \\ 3 & -1 & -1 & 0 & 1 & 1 & 0 \\ \hline 4 & & & 1/n & 1/n & & \\ \hline \end{array} \qquad 10.11$$

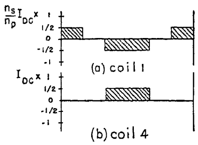


FIG. 10.5. Instantaneous currents.

The currents in coils 1 and 4 are shown in Fig. 10.5. Their r.m.s. values are $nI_{\text{d-c.}}/\sqrt{6}$ and $I_{\text{d-c.}}(\frac{1}{2})\sqrt{(1^2 + 1^2)/6} = I_{\text{d-c.}}/(2\sqrt{3})$.

Twelve-Phase Quadruple Rectifier

Figure 10.6 shows a rectifier connection with thirty-six coils, in which four anodes fire simultaneously. Figure 10.7 shows the number of turns of the various windings ($a = 0.816$ and $b = 0.299$).

The analysis follows the previous one step by step except that here three closed magnetic meshes are assumed and their magnetizing cur-

rents are assumed to be identical and equal to i''' (along axis m) instead of being zero. **C** is given in equation 10.12.

Rows 2 and 3 repeat row 1, also rows 5 and 6 repeat 4.

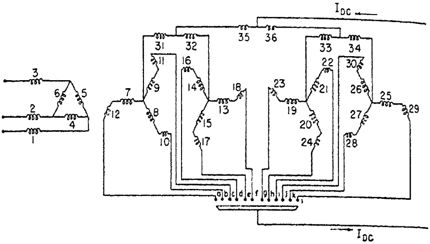


FIG. 10.6. Twelve-phase, quadruple zigzag rectifier.

The instantaneous current in the line (coil 1), primary (coil 4), secondary (coil 7), and interphase reactor (coil 36) are shown in Fig.

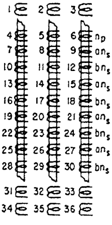


FIG. 10.7. Arrangement of coils.

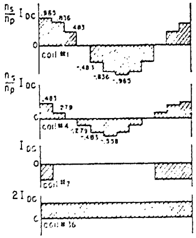


FIG. 10.8. Instantaneous currents in coils 1, 4, 7, and 36.

10.8. The r.m.s. currents are: line = $0.683 I_{d\text{-}c} n_s/n_p$; primary = $0.394 I_{d\text{-}c} n_s/n_p$; and secondary = $(\frac{1}{4}\sqrt{3}) I_{d\text{-}c}$. The magnetizing current flows only in the closed delta.

		a	b	c	d	e	f	g	h	i	j	k	l	m
$\frac{n_s}{n_p}$	1	$4a + 2b$	$3a + 3b$	$2a + b$	0	$-2a - b$	$-3a - 3b$	$-4a - 2b$	$-3a - 3b$	$-2a - b$	0	$2a + b$	$3a + 3b$	0
	2	$-2a - b$	$-3b - 3a$	....	......	....	....	....	....	......	....	.....	......	0
	3	$-2a - b$	0	.....	......	....	....	....	....	......	....	.....	......	0
	4	$2a + b$	$a + b$	0	$-a - b$	$-2a - b$	$-2a - 2b$	$-2a - b$	$-a - b$	0	$a + b$	$2a + b$	$2a + 2b$	$1/n_s$
	5	$-2a - b$	$-2a - 2b$	....	......	....	....	....	....	......	....	.....	......	$1/n_s$
	6	0	$a + b$	....	....	....	....	....	....	......	....	.....	......	$1/n_s$
	7	-1									-1	-1	-1	
	36	2	2	2	2	2	2	2	2	2	2	2	2	

12

EXERCISE

Find **C**, the instantaneous currents, and the r.m.s. value of the currents of the rectifier circuits of Figs. 10.9–10.11.

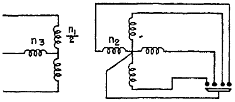


Fig. 10.9. Biphase circuit.

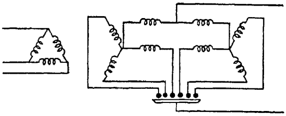


Fig. 10.10. Six-phase double-wye circuit.

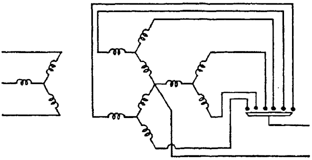


Fig. 10.11. Six-phase forked circuit.

CHAPTER 11

PHASE-SHIFT TRANSFORMERS*

Considering Only One-Third of the Windings

In a *balanced* three-phase system, whatever currents flow in phase a, the same currents, shifted by 120 degrees in time, flow in phase b and by 240 degrees in phase c. That is, if i^a, i^b, and i^c flow in the meshes of phase a, then ai^a, ai^b, and ai^c flow in the meshes of b, and a^2i^a, a^2i^b, and a^2i^c in the meshes of c (see Fig. 11.1 for one mesh per phase).

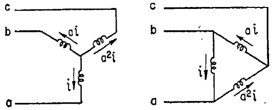


FIG. 11.1. Currents in balanced three-phase circuits.

Hence *in balanced three-phase systems it is sufficient to consider only one-third of the coils, meshes, and currents.*

Representation of Three-Phase Transformers

Let three four-winding transformers be given (Fig. 11.2)

It is customary to represent the coils in the following way.

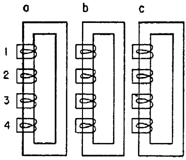


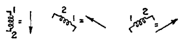
FIG. 11.2. Three-phase four-winding transformers.

1. Those of phase a are always vertical.
2. Those of phase b are always at 120 degrees from it.
3. Those of phase c are always at 240 degrees from it.

* *T.A.N.*, Chapter XIII, p. 339.

Instead of drawing the coil, only a straight line is drawn as shown in Fig. 11.3.

Hence the coils of Fig. 11.2 are represented as shown in Fig. 11.4.



phase a phase b phase c

FIG. 11.3. Representation of phases.

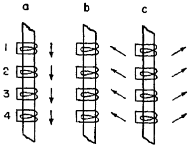
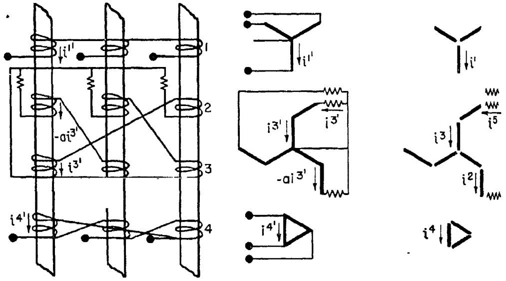


FIG. 11.4. Representation of a four-winding transformer.

Interconnection of Coils

Let the twelve windings of Figs. 11.2 or 11.4 be interconnected as shown in Fig. 11.5. In particular:

1. Windings 1 into star.
2. Windings 3 into star.



(*a*) Actual connections. (*b*) Representation. (*c*) Its primitive.

FIG. 11.5. Phase-shift zigzag transformer.

3. Windings 4 into delta.
4. Windings 2 in opposing series with 3 (the so-called zigzag connection.)
5. Windings 2 are connected in series with a balanced star load.

The Primitive System

It is sufficient to consider the currents only in the first transformer (vertical lines) and in one of the loads.

The primitive network (Fig. 11.2 or 11.5*c*) has five currents

Z =

	1	2	3	4	5
1		Z_{1-2}	Z_{1-3}	Z_{1-4}	
2	Z_{1-2}		Z_{2-3}	Z_{2-4}	
3	Z_{1-3}	Z_{2-3}		Z_{3-4}	
4	Z_{1-4}	Z_{2-4}	Z_{3-4}		
5					Z

i =

1	2	3	4	5
i^1	i^2	i^3	i^4	i^5

e =

1	2	3	4	5
e_1			e_4	

11.1

The Transformation Tensor

In the given network, Fig. 11.5*b*, there are nine meshes; hence it is sufficient to consider *three* meshes and assume *three* new currents in *three of the vertical coils* $i^{1'}$, $i^{3'}$, $i^{4'}$ as shown.

The next step is to determine the currents *in the remaining vertical coils* and in one of the loads.

In Fig. 11.6, if $i^{3'}$ flows from A to B, then $ai^{3'}$ flows from C to B (see Fig. 11.1). Hence $-ai^{3'}$ flows from C to D.

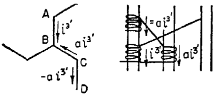


FIG. 11.6. Determining the current in the remaining vertical coil.

Similarly in one of the loads (it does not matter in which one) $i^{3'}$ flows.

Hence, equating the currents flowing in the four vertical coils (and one of the loads) before and after interconnection (comparing Fig. 11.5*b* and *c* and Fig. 11.6),

$$
\begin{aligned}
i^1 &= i^{1'} \\
i^2 &= -ai^{3'} \\
i^3 &= i^{3'} \\
i^4 &= i^{4'} \\
i^5 &= i^{3'}
\end{aligned}
$$

$\mathbf{C}_1$ =

	1′	3′	4′
1	1		
2		$-a$	
3		1	
4			1
5		1	

11.2

The coefficients of the new currents give the transformation tensor **C**,

showing the manner of interconnection of the coils. One of its components is a complex number $-a = 0.5 - j0.866$.

It should be noted that, to establish the currents flowing in all vertical coils, the currents in some of the other coils also had to be established as an intermediary step.

Of course, in place of the coils of phase a (the vertical coils), the coils of any of the other phases could have been considered.

Neglecting Magnetizing Currents*

The procedure from this point is the same as for any other multiwinding transformer network.

The equation of constraint of the vertical coils *before* interconnection is

$$n_1 i^1 + n_2 i^2 + n_3 i^3 + n_4 i^4 = 0 \qquad 11.3$$

Replacing the old currents by the new currents with the aid of equation 11.2,

$$n_1 i^{1'} + n_2 a i^{3'} + n_3 i^{3'} + n_4 i^{4'} = 0 \qquad 11.4$$

Assuming, say, $i^{4'}$ (the current in the delta) as the dependent current,

$$\begin{aligned} i^{1'} &= i^{1'} \\ i^{3'} &= i^{3'} \\ i^{4'} &= -\frac{n_1}{n_4} i^{1'} + \left(\frac{n_2 a}{n_4} - \frac{n_3}{n_4}\right) i^{3'} \end{aligned} \qquad \mathbf{C}_2 = \begin{array}{c|cc} & 1'' & 3'' \\ \hline 1' & 1 & \\ 3' & & 1 \\ 4' & N_1 & N_2 \end{array} \qquad 11.5$$

Note that N_2 is a complex number.

$$\mathbf{C} = \mathbf{C}_1 \cdot \mathbf{C}_2 = \begin{array}{c|cc} & 1'' & 3'' \\ \hline 1 & 1 & \\ 2 & & -a \\ 3 & & 1 \\ 4 & N_1 & N_2 \\ 5 & & 1 \end{array} \qquad \mathbf{C}_t^* = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1'' & 1 & & & N_1^* & \\ 3'' & & -a^2 & & N_2^* & \end{array} \qquad 11.6$$

* *G.E.R.*, May, 1935, p. 237.

Currents and Differences of Potentials

$$
\mathbf{C}_t^* \cdot \mathbf{Z} \cdot \mathbf{C} = \mathbf{Z}' = \begin{array}{c|c|c|}
 & 1'' & 3'' \\
\hline
1'' & Z_{1-4}(N_1 + N_1^*) & \begin{array}{l} -aZ_{1-2} + Z_{1-3} + N_2 Z_{1-4} \\ -aN_1^* Z_{2-4} + N_1^* Z_{3-4} \end{array} \\
\hline
3'' & \begin{array}{l} -a^2 Z_{1-2} + Z_{1-3} + N_2^* Z_{1-4} \\ -a^2 N_1 Z_{2-4} + N_1 Z_{3-4} \end{array} & \begin{array}{l} Z_{2-3} - Z_{2-4}(N_2 a^2 + N_2^* a) \\ + Z_{3-4}(N_2 + N_2^*) + Z \end{array} \\
\hline
\end{array} \qquad 11.7
$$

$$
\mathbf{e}' = \mathbf{C}_t^* \cdot \mathbf{e} = \begin{array}{|c|c|}
\multicolumn{1}{c}{1''} & \multicolumn{1}{c}{3''} \\
\hline
e_1 + N_1^* e_4 & N_2^* e_4 \\
\hline
\end{array} \qquad 11.8
$$

The currents are from $\mathbf{i}' = \mathbf{Z}'^{-1} \cdot \mathbf{e}$

$$
\mathbf{i}' = \begin{array}{|c|c|}
\multicolumn{1}{c}{1''} & \multicolumn{1}{c}{3''} \\
\hline
i^{1''} & i^{3''} \\
\hline
\end{array} \qquad 11.9
$$

In each coil of the first transformer (vertical coils) flows

$$
\mathbf{i}_c = \mathbf{C} \cdot \mathbf{i}' = \begin{array}{|c|c|c|c|c|}
\multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{4} & \multicolumn{1}{c}{5} \\
\hline
i^{3''} & -ai^{3''} & i^{3''} & N_1 i^{a''} + N_2 i^{3''} & i^{3''} \\
\hline
\end{array} \qquad 11.10
$$

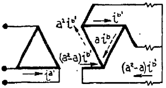
The differences of potential across each coil of the first transformer are

$$
\mathbf{e}_c = \mathbf{Z} \cdot \mathbf{C} \cdot \mathbf{i}' = \begin{array}{c|l|}
\hline
1 & N_1 Z_{1-4} i^{1''} + (-aZ_{1-2} + Z_{1-3} + N_2 Z_{1-4}) i^{3''} \\
\hline
2 & (Z_{1-2} + N_1 Z_{2-4}) i^{1''} + (Z_{2-3} + N_2 Z_{2-4}) i^{3''} \\
\hline
3 & (Z_{1-3} + N_1 Z_{3-4}) i^{1''} + (-aZ_{2-3} + N_2 Z_{3-4}) i^{3''} \\
\hline
4 & Z_{1-4} i^{1''} + (-aZ_{2-4} + Z_{3-4}) i^{3''} \\
\hline
5 & \multicolumn{1}{c|}{Z i^{3''}} \\
\hline
\end{array} \qquad 11.11
$$

TABLE IV

Balanced Three-Phase Multiwinding Transformers and Their Transformation Matrices

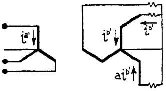
First column—C_1 shows interconnection of coils.
Second column—C_2 neglects magnetizing currents.
Third column—C_3 represents their resultant.



$$C_1 = \begin{array}{c|cc} & a' & b' \\ \hline a & 1 & \\ b & & 1 \\ c & & a^2-a \\ 1 & & a^2-a \end{array} \qquad C_2 = \begin{array}{c|c} & b'' \\ \hline a' & -N \\ b' & 1 \end{array} \qquad C = \begin{array}{c|c} & b'' \\ \hline a & -N \\ b & 1 \\ c & a^2-a \\ 1 & a^2-a \end{array}$$

$$N = \frac{n_b + (a^2-a)n_c}{n_a}$$

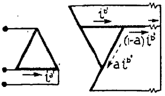
STUB DELTA



$$C_1 = \begin{array}{c|cc} & a' & b' \\ \hline a & 1 & \\ b & & 1 \\ c & & -a \\ 1 & & 1 \end{array} \qquad C_2 = \begin{array}{c|c} & b'' \\ \hline a' & -N \\ b' & 1 \end{array} \qquad C = \begin{array}{c|c} & b'' \\ \hline a & -N \\ b & 1 \\ c & -a \\ 1 & 1 \end{array}$$

$$N = \frac{n_b - a n_c}{n_a}$$

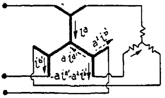
ZIG-ZAG



$$C_1 = \begin{array}{c|cc} & a' & b' \\ \hline a & 1 & \\ b & & 1 \\ c & & 1-a \\ 1 & & 1-a \end{array} \qquad C_2 = \begin{array}{c|c} & b'' \\ \hline a' & -N \\ b' & 1 \end{array} \qquad C = \begin{array}{c|c} & b'' \\ \hline a & -N \\ b & 1 \\ c & 1-a \\ 1 & 1-a \end{array}$$

$$N = \frac{n_b + (1-a)n_c}{n_a}$$

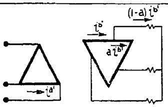
EXTENDED DELTA



$$C_1 = \begin{array}{c|cc} & a' & b' \\ \hline a & 1 & \\ b & & 1 \\ c & -a & -a^2 \\ 1 & -a & -a^2 \end{array} \qquad C_2 = \begin{array}{c|c} & b'' \\ \hline a' & -N \\ b' & 1 \end{array} \qquad C = \begin{array}{c|c} & b'' \\ \hline a & -N \\ b & 1 \\ c & aN-a^2 \\ 1 & aN-a^2 \end{array}$$

$$N = \frac{n_b - a^2 n_c}{n_a - a n_c}$$

DOUBLE ZIG-ZAG



$$C_1 = \begin{array}{c|cc} & a' & b' \\ \hline a & 1 & \\ b & & 1 \\ c & & a \\ 1 & & 1-a \end{array} \qquad C_2 = \begin{array}{c|c} & b'' \\ \hline a' & -N \\ b' & 1 \end{array} \qquad C = \begin{array}{c|c} & b'' \\ \hline a & -N \\ b & 1 \\ c & a \\ 1 & 1-a \end{array}$$

$$N = \frac{n_b + a n_c}{n_a}$$

INSCRIBED DELTA

EXERCISES

1. Find C_1 and C_2 of the zigzag transformer of Fig. 11.7 (consisting of three three-winding transformers).

2. Find C_1 and C_2 of the inscribed delta transformer of Fig. 11.8 (three two-winding transformers.

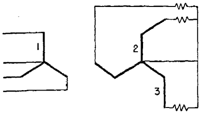


FIG. 11.7. Zigzag transformer.

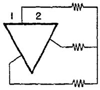


FIG. 11.8. Inscribed delta transformer.

3. Find C_1 and C_2 of the zigzag auto-transformer of Fig. 11.9 (three three-winding transformers).

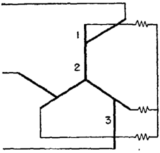


FIG. 11.9. Zigzag auto-transformer.

4. Find the currents and differences of potentials across the coils of the transformers shown in Table IV.

CHAPTER 12

INDEX NOTATION*

Denoting the Reference Axes of a Particular Frame

(*a*) In the notation hitherto used (the "direct" notation), each physical entity (tensor) was denoted by a single symbol **e** or **Z** (the "base" letter) without showing its valence or its law of transformation. Also the notation could not restrict the analysis so that it should apply to only one portion of the network. The "index notation" to be shown now takes care of these and other needs of the analysis.

The symbols $a, b, c \cdots$ denoting the individual reference axes will be called "fixed" indices. The totality of all axes will be denoted by "α" or "β" to be called "variable" indices, so that *α assumes all the fixed indices in succession.*

The base letter of a vector (tensor of valence 1) will have one variable index as e_α

$$\begin{array}{c|c|c|c|c|} \alpha & a & b & c & d \\ \hline e_\alpha = & 5 & 3 & 2 & 1 \\ \hline \end{array}$$

so that $e_b = 3$, $e_d = 1$.

The base letter of a tensor of valence 2 will have two indices

$$Z_{\alpha\beta} = \begin{array}{c|c|c|c|} \alpha \downarrow \;\; \beta \rightarrow & a & b & c \\ \hline a & 1 & 2 & 3 \\ \hline b & 4 & 5 & 6 \\ \hline c & 7 & 8 & 9 \\ \hline \end{array} = \begin{array}{c|c|c|c|} \alpha \backslash \beta & a & b & c \\ \hline a & 1 & 2 & 3 \\ \hline b & 4 & 5 & 6 \\ \hline c & 7 & 8 & 9 \\ \hline \end{array}$$

so that $Z_{ac} = 3$, $Z_{bb} = 5$, etc.

The transpose of $A_{\alpha\beta}$ is $A_{\beta\alpha}$. The inverse of $A_{\alpha\beta}$ is denoted by a *different base letter*, as

$$(A_{\alpha\beta})^{-1} = B^{\beta\alpha} \qquad 12.1$$

* *T.A.N.*, Chapter VII.

Note that the order and position of the indices are interchanged. $(Y^{\alpha\beta})^{-1} = Z_{\beta\alpha}$ and $(Z_{\alpha\beta})^{-1} = Y^{\beta\alpha}$.

(*b*) By using both variable and fixed indices, any row or column may be picked out of a tensor at will. For instance, $Z_{\alpha b}$ is the second column, $Z_{c\beta}$ is the third row.

A tensor of valence 3 has three indices as shown in Fig. 12.1.

A tensor of valence 0 (scalar) has no index.

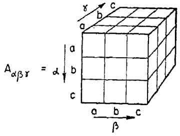


FIG. 12.1.

Denoting the Various Reference Frames

The various reference frames are usually denoted by priming the indices as $Z_{\alpha'\beta'}, Z_{\alpha''\beta''}, Z_{\alpha'''\beta'''} \cdots$. In general, $Z_{\alpha\beta}$ stands for *all* the possible reference frames (primes, double primes, triple primes, etc.) in addition to all components Z_{aa}, Z_{ab}, etc.

The symbol Z, the base letter, still represents the whole physical entity, while the indices show just which reference frame and in that frame just which component or components are under consideration. For instance, for the tensor $Z_{\alpha\beta}$

$$Z_{\alpha'\beta'} = \begin{array}{c|cc} {}_{\alpha'}\diagdown^{\beta'} & a' & b' \\ \hline a' & 1 & 2 \\ b' & 3 & 4 \end{array} \qquad Z_{\alpha''\beta''} = \begin{array}{c|cc} {}_{\alpha''}\diagdown^{\beta''} & a'' & b'' \\ \hline a'' & 5 & 6 \\ b'' & 7 & 8 \end{array} \qquad Z_{\alpha'''\beta'''} = \begin{array}{c|cc} {}_{\alpha'''}\diagdown^{\beta'''} & a''' & b''' \\ \hline a''' & 9 & 0 \\ b''' & -2 & -5 \end{array}$$

Also

$$Z_{a''b''} = 6, \quad Z_{b'''\beta'''} = \begin{array}{|c|c|} \hline -2 & -5 \\ \hline \end{array}$$

When the indices contain no primes and are all variable indices as $Z_{\alpha\beta}$, then (if not otherwise stated) they imply all the components of all possible reference frames, that is, the entity itself.

It is customary to use different variable indices for different reference frames (instead of different numbers of primes) such as $Z_{\alpha\beta}$ for one,

Z_{mn} for another frame. Though this notation is permissible, it is not logical, unless the two reference frames are of different types.

Denoting the Manipulations to be Performed

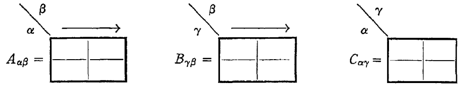
(*a*) The rule for multiplying together two vectors $\mathbf{e} \cdot \mathbf{i}$ (each with, say, four fixed indices a, b, c, d) can be represented with the aid of indices as

$$\begin{array}{c|c|c|c|c|} & a & b & c & d \\ \cline{2-5} \mathbf{e} = & 1 & 2 & 3 & 4 \\ \cline{2-5} \end{array} \qquad \begin{array}{c|c|c|c|c|} & a & b & c & d \\ \cline{2-5} \mathbf{i} = & 5 & 6 & 7 & 8 \\ \cline{2-5} \end{array}$$

$$\mathbf{e} \cdot \mathbf{i} = \sum_{\alpha=a}^{d} e_\alpha i_\alpha = 1 \times 5 + 2 \times 6 + 3 \times 7 + 4 \times 8 = 70 \qquad 12.2$$

Similarly the rule for multiplying together two tensors of valence 2 is represented as

$$\sum_{\beta=a}^{d} A_{\alpha\beta} B_{\gamma\beta} = C_{\alpha\gamma} \qquad 12.3$$



(*b*) The index β, along which the summation is performed, is called the "dummy index." *The arrows are always drawn along the dummy indices.* The remaining indices α and γ are called "free indices." The resultant tensor $C_{\alpha\gamma}$ contains only the free indices. E.g.,

$$\sum_{\gamma} A_{\alpha\beta\gamma} B_{\gamma\delta} = C_{\alpha\beta\delta} \qquad 12.4$$

showing that the resultant of the product is a tensor of valence 3. It may be said that the two dummy indices stand for a dot-product. (In direct notation only $\mathbf{A} \cdot \mathbf{B} = A_{\alpha\beta} B_{\beta\gamma}$ can be represented, but not $A_{\alpha\beta} B_{\gamma\beta}$.)

(*c*) *Since the dummy index occurs twice, the summation sign may be left out* as

$$A_{\alpha\beta\gamma} B_{\gamma\delta} = C_{\alpha\beta\delta} \qquad 12.5$$

This is called the "Einstein convention" as he was the first to suggest the omission of the summation sign.

Denoting the Law of Transformation

(*a*) Although the currents $\mathbf{i}$ and voltages $\mathbf{e}$ are both vectors, they have different laws of transformation

$$\mathbf{i} = \mathbf{C}\cdot\mathbf{i}' \quad \text{and} \quad \mathbf{e} = \mathbf{C}_t^{-1}\cdot\mathbf{e}' \qquad 12.6$$

one attracting $\mathbf{C}$, the other $\mathbf{C}_t^{-1}$.

In other words, $\mathbf{e}$ *and* $\mathbf{i}$ *are two different types of vectors as they behave differently when coils are interconnected into various networks* (or even when the reference frame is changed and the network is left undisturbed). That is, when coils are connected in series, the voltages are added, but the currents remain unchanged. On the other hand, when coils are connected in parallel, the voltages now remain unchanged and the currents are added.

To represent this difference in their physical behavior, hence in their law of transformation, the current vector always has an "upper" or "contravariant" index as i^α, while the voltage vector always has a "lower" or "covariant" index as e_α. Also i^α is called a "contravariant" vector and e_α a "covariant" vector.

(*b*) In general, *if an old index attracts* $\mathbf{C}^{-1}$ (*or* $\mathbf{C}_t^{-1}$), *it is an upper index; if it attracts* $\mathbf{C}$ (*or* $\mathbf{C}_t$), *it is a lower index.*

Since $\mathbf{Z}$ attracts $\mathbf{C}$ twice, equation 6.6, both its indices are lower indices as $Z_{\alpha\beta}$. However, $\mathbf{Y}$ attracts $\mathbf{C}^{-1}$ twice, equation 6.8; hence it is written as $Y^{\alpha\beta}$. On the other hand $\mathbf{C}$ itself attracts one $\mathbf{C}$ and one $\mathbf{C}^{-1}$, equation 6.11; hence it has one upper and one lower index as $C^\alpha_{\alpha'}$. Its inverse, $\mathbf{C}^{-1}$, is written $C^{\alpha'}_\alpha$.

Tensors may have any number of covariant and contravariant indices as $A_{\alpha\cdot\gamma\cdot\cdot}^{\cdot\beta\cdot\delta}$. So that no confusion may arise as to the order of the indices, dots are placed in the empty positions.

The only exceptions in disregarding the order of the indices are the transformation tensor $C^\alpha_{\alpha'}$ and the unit tensor $I^\alpha_{\alpha'}$.

The Dummy-Index Rule

It so happens in nature that every type of energy is the product of two vectors, one being always a "covariant" vector, the other a "contravariant" vector. For instance, $T = \varphi_m i^m$ or $T = M_m v^m$ (M_m = momentum). Similarly, with power, $P = e_m i^m$.

In general, in any problems of tensor analysis, *of the two dummy indices one is always a lower, the other an upper index.*

With the aid of this rule, the law of transformation of any tensor may be written down automatically

$$\begin{array}{l|l} i^{\alpha} = i^{\alpha'} C^{\alpha}_{\alpha'} & Z_{\alpha'\beta'} = Z_{\alpha\beta} C^{\alpha}_{\alpha'} C^{\beta}_{\beta'} \\ e_{\alpha} = e_{\alpha'} C^{\alpha'}_{\alpha} & Y^{\alpha'\beta'} = Y^{\alpha\beta} C^{\alpha'}_{\alpha} C^{\beta'}_{\beta} \end{array} \qquad 12.7$$

Tensor Equations

(*a*) Every term in a tensor equation must have the same free indices. With one free index every term is a vector, as

$$e_{\alpha} = R_{\alpha\beta} i^{\beta} + L_{\alpha\beta} \frac{di^{\beta}}{dt} + \Gamma_{\alpha\beta\gamma} i^{\beta} i^{\gamma} \qquad 12.8$$

In every term the free index is α. This equation stands for n ordinary equations in every reference frame.

With two free indices every term is a tensor of valence 2.

$$Z'_{\alpha\beta} = Z_{\alpha\beta} - Z_{\alpha\gamma} Y^{\gamma\delta} Z_{\delta\beta} \qquad 12.9$$

This tensor equation stands for n^2 ordinary equations in every reference frame.

With no free indices every term is a scalar.

$$P = e_{\alpha} i^{\alpha} \qquad 12.10$$

This tensor equation stands for one ordinary equation in every reference frame.

(*b*) The dummy indices may be changed in each term at will.

$$R_{\alpha\beta} i^{\beta} = R_{\alpha\gamma} i^{\gamma} \qquad 12.11$$

The free indices may be changed only in all the terms of an equation at the same time.

$e_{\alpha} = R_{\alpha\beta} i^{\beta}$ may be written as $e_{\gamma} = R_{\gamma\beta} i^{\beta}$.

(*c*) With index notation the order of the tensors in a product can be changed at will

$$A_{\alpha\beta} B^{\beta\gamma} = B^{\beta\gamma} A_{\alpha\beta} \quad \Big| \quad \text{but} \quad \mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A} \qquad 12.12$$

The dummy index β shows whether the arrows are drawn horizontally or vertically. However, if the components of a tensor contain operators, such as $p = d/dt$, their order cannot be changed (just as the order of p in an ordinary equation cannot be changed).

Contraction

It has been assumed hitherto that the two dummy indices occur in different tensors as $Z_{\alpha\beta} i^{\beta}$. However, they may occur in the same tensor.

Let

$$A^{\alpha}_{\cdot\beta} = \begin{array}{c|ccc} {}_{\alpha}\diagdown^{\beta} & a & b & c \\ \hline a & 2 & 1 & 5 \\ b & 9 & 3 & 6 \\ c & 2 & 8 & 4 \end{array}$$

Then

$$A^{\alpha}_{\cdot\alpha} = A^{a}_{\cdot a} + A^{b}_{\cdot b} + A^{c}_{\cdot c} = 2 + 3 + 4 = 9 \qquad 12.13$$

That is, $A^{\alpha}_{\cdot\alpha}$ *represents the sum of the diagonal terms.*

In general, assuming two dummy indices in a tensor (the process of "contraction") lowers its valence by 2. If $K^{\cdot\cdot\cdot\delta}_{\alpha\beta\gamma}$ is a tensor of valence 4, then $K^{\cdot\cdot\cdot\gamma}_{\alpha\beta\gamma}$ is a tensor of valence 2, namely $K_{\alpha\beta}$.

EXERCISES

1. How are each of the shaded portions of Fig. 12.2 represented in index notation?
2. What is the law of transformation of the tensor $K^{\cdot\cdot\cdot\delta}_{\alpha\beta\gamma\cdot}$?

$A_{\alpha\beta\gamma}$ =

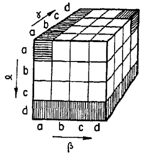


FIG. 12.2.

3. What is wrong with the following equations?

$$(a)\ e_\alpha = R_{\beta\gamma} i^\alpha; \quad (b)\ A_{\alpha\beta} = B^{\cdot\delta}_{\alpha\cdot} C^{\delta}_{\cdot\beta}; \quad (c)\ A^{\alpha}_{\cdot\beta} = B_{\alpha\beta\gamma} i^\gamma.$$

4. Is the following equation correctly written (that is, are the indices correctly balanced)?

$$e_\alpha = R_{\alpha\beta} i^\beta + a_{\alpha\beta} \frac{di^\beta}{dt} + \Gamma_{\beta\gamma\alpha} i^\beta i^\gamma$$

5. Write out all the four sets of equations 14.7 as shown in equation 14.8.

CHAPTER 13

DIFFERENTIATION AND INTEGRATION OF TENSORS*

Differentiation

The differentiation and integration of tensors are facilitated by the use of the index notation, as the indices show the succession of steps to be performed. The following rules also apply to n-way matrices.

(*a*) A tensor is differentiated with respect to a scalar by differentiating each of its components separately in every reference frame. The valence of the tensor remains unchanged.

$$A_{\alpha\beta} = \begin{bmatrix} 2 & & -7 \\ & \cos\theta & -\sin\theta \\ \theta^2 & & -\theta \end{bmatrix} \qquad \frac{\partial A_{\alpha\beta}}{\partial\theta} = B_{\alpha\beta} = \begin{bmatrix} & & \\ & -\sin\theta & -\cos\theta \\ 2\theta & & -1 \end{bmatrix} \tag{13.1}$$

(*b*) A tensor is differentiated with respect to a *vector* x^α by differentiating each component of the tensor with respect to each component of the vector *in succession*. The valence of the new tensor is one larger. For instance, find $\partial A_\alpha/\partial x^\beta$, where

$$A_\alpha = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} x^2yu & yz & x^2z^2 & u^3 \end{bmatrix} \end{array} \qquad x^\beta = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} x & y & z & u \end{bmatrix} \end{array}$$

$$\beta = a \cdots \frac{\partial A_\alpha}{\partial x} = a \quad \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 2xyu & & 2xz^2 & \end{bmatrix} \end{array}$$

$$\beta = b \cdots \frac{\partial A_\alpha}{\partial y} = b \quad \begin{bmatrix} x^2u & z & & \end{bmatrix}$$

$$\beta = c \cdots \frac{\partial A_\alpha}{\partial z} = c \quad \begin{bmatrix} & y & 2zx^2 & \end{bmatrix}$$

$$\beta = d = \frac{\partial A_\alpha}{\partial u} = d \quad \begin{bmatrix} x^2y & & & 3u \end{bmatrix}$$

$$\frac{\partial A_\alpha}{\partial x^\beta} = \begin{array}{c|cccc} \beta \backslash \alpha & a & b & c & d \\ \hline a & 2xyu & & 2xz^2 & \\ b & x^2u & z & & \\ c & & y & 2zx^2 & \\ d & x^2y & & & 3u \end{array} \tag{13.2}$$

* *T.A.N.*, Chapter I, p. 31.

The expression $\partial A_\alpha / \partial x^\beta$ is denoted as $B_{\alpha\beta}$. That is, *the contravariant (upper) index β in the denominator becomes a covariant (lower) index in the resultant tensor.*

$$\frac{\partial K_{\alpha\cdot\gamma}^{\cdot\beta}}{\partial A_{\cdot\epsilon}^{\delta}} = M_{\alpha\cdot\gamma\delta\cdot}^{\cdot\beta\cdot\cdot\epsilon} \tag{13.3}$$

(*c*) A product of tensors is differentiated by differentiating each tensor separately. For instance,

$$\frac{\partial(A_{\alpha\beta}B^{\beta\gamma})}{\partial x^\delta} = \frac{\partial A_{\alpha\beta}}{\partial x^\delta} B^{\beta\gamma} + A_{\alpha\beta} \frac{\partial B^{\beta\gamma}}{\partial x^\delta} \tag{13.4}$$

Gradient

In physical problems three types of differentiation occur rather frequently.

The derivative of a tensor with respect to a vector x^α is called the "gradient" of a tensor.

$$\begin{aligned} \text{Grad } A &= \frac{\partial A}{\partial x^\alpha} = A_\alpha \\ \text{Grad } B_\alpha &= \frac{\partial B_\alpha}{\partial x^\beta} = C_{\alpha\beta} \end{aligned} \tag{13.5}$$

Not only a scalar but also a tensor of any valence may have a gradient. The valence of the gradient is *one greater* than that of the original tensor.

From the gradient, the divergent and the curl are built up in the following manner.

Divergent

If the gradient of a tensor is "contracted," the resultant is called the "divergent" of the tensor.

$$\begin{aligned} \text{Div } A_\alpha &= \frac{\partial A_\alpha}{\partial x^\alpha} = B \\ \text{Div } C_{\alpha\beta} &= \frac{\partial C_{\alpha\beta}}{\partial x^\beta} = D_\alpha \end{aligned} \tag{13.6}$$

Not only a vector but also a tensor of any valence may have a divergent. The valence of the divergent is *one less* than that of the original tensor. E.g.,

α	a	b	c
$A_\alpha =$	x^2	xy	xyz

α	a	b	c
$x^\alpha =$	x	y	z

$$\text{Div } A_\alpha = \frac{\partial A_\alpha}{\partial x^\alpha} = \frac{\partial A_a}{\partial x^a} + \frac{\partial A_b}{\partial x^b} + \frac{\partial A_c}{\partial x^c} = 2x + x + xy \tag{13.7}$$

The same result could have also been found by calculating first the gradient, that is $\partial A_\alpha/\partial x^\beta = B_{\alpha\beta}$, then adding its diagonal components.

Curl

When the gradient of a tensor has been calculated, then, if its transpose is subtracted, the resultant is the "curl" of the original tensor.

$$\text{Curl}\, A_\alpha = \frac{\partial A_\alpha}{\partial x^\beta} - \frac{\partial A_\beta}{\partial x^\alpha} = B_{\alpha\beta} - B_{\beta\alpha} \qquad 13.8$$

That is, if $\partial A_\alpha/\partial x^\beta = B_{\alpha\beta}$, then its transpose is $B_{\beta\alpha}$.

$$\text{Curl}\, C_{\alpha\beta} = \frac{\partial C_{\alpha\beta}}{\partial x^\gamma} - \frac{\partial C_{\alpha\gamma}}{\partial x^\beta} = D_{\alpha\beta\gamma} - D_{\alpha\gamma\beta} \qquad 13.9$$

Not only a vector but also a tensor of any valence may have a curl. The valence of the curl is *one greater* than that of the original tensor. For instance,

$$A_\alpha = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \boxed{\begin{array}{c|c|c|c} x^2y & yz & x^2z^2 & 3u^2 \end{array}} \end{array} \qquad x^\beta = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \boxed{\begin{array}{c|c|c|c} x & y & z & u \end{array}} \end{array}$$

$$\frac{\partial A_\alpha}{\partial x^\beta} = \left[\begin{array}{c|c|c|c} 2xyu & 0 & 2xz^2 & 0 \\ \hline x^2u & z & 0 & 0 \\ \hline 0 & y & 2zx^2 & 0 \\ \hline x^2y & 0 & 0 & 3u \end{array}\right] \qquad \frac{\partial A_\beta}{\partial x^\alpha} = \left[\begin{array}{c|c|c|c} 2xyu & x^2u & 0 & x^2y \\ \hline 0 & z & y & 0 \\ \hline 2xz^2 & 0 & 2zx^2 & 0 \\ \hline 0 & 0 & 0 & 3u \end{array}\right]$$

$$\text{Curl}\, A_\alpha = \frac{\partial A_\alpha}{\partial x^\beta} - \frac{\partial A_\beta}{\partial x^\alpha} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & -x^2u & 2xz^2 & -x^2y \\ b & x^2u & 0 & -y & 0 \\ c & -2xz^2 & y & 0 & 0 \\ d & x^2y & 0 & 0 & 0 \end{array} \qquad 13.10$$

The above tensor is "skew symmetric," that is, all terms to the right of the main diagonal line are negative to those to the left. The diagonal terms are zero. Hence the number of different components is $n^2/2 - n$. This skew symmetry of the curl exists in every reference frame.

Line, Surface, and Volume Integrals

(*a*) A tensor of any rank is integrated with respect to a scalar by integrating each of its components.

$$A_\alpha = \begin{array}{c|c|c|c} a & b & c & d \\ \hline \sin\theta & \cos\theta & 3 & -\sin\theta \end{array}$$

$$\int A_\alpha d\theta = B_\alpha = \begin{array}{c|c|c|c} a & b & c & d \\ \hline -\cos\theta + A & \sin\theta + B & 3\theta + C & \cos\theta + D \end{array} \qquad 13.11$$

(*b*) A tensor is integrated with respect to a vector by integrating each component of the tensor with respect to each component of the vector and performing the contraction as indicated by the indices. For instance, if

$$A_\alpha = \begin{array}{c|c|c|c} a & b & c & d \\ \hline \sin x & \cos y & 3 & 2 \end{array} \qquad dx^\alpha = \begin{array}{c|c|c|c} a & b & c & d \\ \hline dx & dy & dz & du \end{array}$$

$$\begin{aligned} \int A_\alpha\, dx^\alpha = k &= \int A_a\, dx^a + \int A_b\, dx^b + \int A_c\, dx^c + \int A_d\, dx^d \\ &= \int \sin x\, dx + \int \cos y\, dy + \int 3dz + \int 2du \\ &= -(\cos x + A) + (\sin y + B) + (3z + C) \\ &\quad + (2u + D) \end{aligned} \qquad 13.12$$

Also

$$\int A^{\alpha}_{\cdot\beta}\, dx^\beta = B^\alpha$$

Such integrals are called "line integrals."

(*c*) The differentials may form a tensor of valence 2 (representing a surface). In that case the contraction is performed twice in succession.

$$\iint A^{\alpha}_{\cdot\beta\gamma}\, dx^\beta\, dx^\gamma = \iint A^{\alpha}_{\cdot\beta\gamma}\, dB^{\beta\gamma} = C^\alpha \qquad 13.13$$

Such integrals are called "surface integrals."

(*d*) "Volume integrals" assume the following form:

$$\iiint A^{\alpha\cdots\cdot}_{\cdot\beta\gamma\delta\epsilon}\, dx^\gamma\, dx^\delta\, dx^\epsilon = \iiint A^{\alpha\cdots\cdot}_{\cdot\beta\gamma\delta\epsilon}\, dB^{\gamma\delta\epsilon} = C^{\alpha}_{\cdot\beta} \qquad 13.14$$

Stokes' Theorem

In tensor analysis the theorem that "the line integral of a vector is equal to the surface integral of the curl of the vector" assumes the form

$$\int A_\alpha\, dx^\alpha = \iint \left(\frac{\partial A_\alpha}{\partial x^\beta} - \frac{\partial A_\beta}{\partial x^\alpha}\right) dx^\alpha\, dx^\beta \qquad 13.15$$

the indices indicating the routine steps that have to be performed in the integration.

Of course, α may have more than three fixed indices and A_α may be replaced by a tensor of any valence. Tensor analysis also supplies a routine procedure for the cases when the reference frames are not rectilinear but curvilinear.

EXERCISES

1\. If

$A_\alpha =$

α	a	b	c	d
A_α	xy^2	yz^2	zu^2	ux^2

$x^\alpha =$

α	a	b	c	d
x^α	x	y	z	u

$A_{\alpha\beta} =$

$\beta \backslash \alpha$	a	b	c	d
a	x^2		xy	
b		y^2		$2u$
c			$-z^2$	
d	$-z$			u^2

Find: (*a*) $\dfrac{\partial A_\alpha}{\partial y}$; (*b*) $\dfrac{\partial A_\alpha}{\partial x^\beta}$; (*c*) $\dfrac{\partial A_{\alpha\beta}}{\partial z}$; (*d*) $\dfrac{\partial A_{\alpha\beta}}{\partial x^\gamma}$.

2\. (*a*) $\dfrac{\partial(A_\alpha B_\beta)}{\partial x^\gamma}$; (*b*) $\dfrac{\partial(A_{\alpha\beta}B^\beta)}{\partial x^\gamma}$; (*c*) $\dfrac{\partial A_{\alpha\beta}B^\alpha C^\beta}{\partial x^\gamma}$.

3\. Find the gradient, divergent, and curl of A_α and $A_{\alpha\beta}$ of exercise 1.

4\. Find the line integral of A_α and the surface integral of $A_{\alpha\beta}$ of exercise 1.

CHAPTER 14

THE FIELD EQUATIONS OF MAXWELL*

Three-Dimensional Form of the Equations

Important examples for the differentiation of tensors are the field equations of Maxwell. In the symbolism of conventional vector analysis they are as follows:

The first set of the field equations (in Heaviside-Lorentz units) is

$$\left.\begin{aligned} \text{Curl}\,\mathbf{H} &= \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{\rho \mathbf{v}}{c} \\ \text{Div}\,\mathbf{D} &= \rho \end{aligned}\right\} \quad \text{I} \qquad 14.1$$

The second set is

$$\left.\begin{aligned} \text{Curl}\,\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0 \\ \text{Div}\,\mathbf{B} &= 0 \end{aligned}\right\} \quad \text{II} \qquad 14.2$$

where ρ satisfies the equation of continuity

$$\text{Div}\,\rho\mathbf{v} + \frac{\partial \rho}{\partial t} = 0 \qquad \text{III} \qquad 14.3$$

and **E** and **B** are expressible in terms of the scalar potential φ and vector potential **A**

$$\left.\begin{aligned} \mathbf{E} &= -\,\text{grad}\,\varphi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \text{curl}\,\mathbf{A} \end{aligned}\right\} \quad \text{IV} \qquad 14.4$$

The vectors have the form

$$\mathbf{E} = \begin{array}{ccc} \text{x} & \text{y} & \text{z} \\ \hline \multicolumn{1}{|c|}{E_x} & \multicolumn{1}{c|}{E_y} & \multicolumn{1}{c|}{E_z} \\ \hline \end{array} \qquad \mathbf{v} = \begin{array}{ccc} \text{x} & \text{y} & \text{z} \\ \hline \multicolumn{1}{|c|}{v_x} & \multicolumn{1}{c|}{v_y} & \multicolumn{1}{c|}{v_z} \\ \hline \end{array} \qquad 14.5$$

* See, for instance, Becker, "Theory der Elektrizität," Vol. II, Teubner, Leipzig, 1933.

Four-Dimensional Tensors

The three-dimensional forms of Maxwell's equations have the following limitations:

1. They are not valid if the reference axes have accelerated motion such as rotation.
2. Even when the reference axes are stationary, the equations are not valid if the velocity **v** of the charges approaches that of light.
3. Unless the reference axes are orthogonal, the calculation of gradient, divergent, and curl becomes rather involved.

All these limitations are removed if the above equations are restated in the language of tensor analysis. Assuming a *rectangular* reference frame along x, y, z (the "primitive" reference frame), Minkowsky gave the following tensor forms of Maxwell's equations, each replacing a set of two conventional vector equations.

First let new types of tensors be introduced by augmenting the three space directions x, y, z by a fourth, the time t. These new tensors are

$$x^\alpha = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline x & y & z & jct \end{array} \qquad \varphi_\alpha = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline A_x & A_y & A_z & j\varphi \end{array}$$

$$s_\alpha = 4\pi \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline \dfrac{\rho_0 V^x}{\beta c} & \dfrac{\rho_0 V^y}{\beta c} & \dfrac{\rho_0 V^z}{\beta c} & \dfrac{j\rho_0}{\beta} \end{array}$$

$$F_{\alpha\beta} = \begin{array}{c|cccc} {}_\alpha\backslash^\beta & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & B_z & -B_y & -jE_x \\ 2 & -B_z & 0 & B_x & -jE_y \\ 3 & B_y & -B_x & 0 & -jE_z \\ 4 & jE_x & jE_y & jE_z & 0 \end{array} \tag{14.6}$$

$$\overset{*}{F}{}^{\alpha\beta} = \begin{array}{c|cccc} {}_\alpha\backslash^\beta & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & E_z & -E_y & jB_x \\ 2 & -E_z & 0 & E_x & jB_y \\ 3 & E_y & -E_x & 0 & jB_z \\ 4 & -jB_x & -jB_y & -jB_z & 0 \end{array} \qquad H^{\alpha\beta} = \begin{array}{c|cccc} {}_\alpha\backslash^\beta & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & H^z & -H^y & -jD^x \\ 2 & -H^z & 0 & H^x & -jD^y \\ 3 & H^y & -H^x & 0 & -jD^z \\ 4 & jD^x & jD^y & jD^z & 0 \end{array}$$

where $F_{\alpha\beta}$ and $H^{\alpha\beta}$ are *skew-symmetric* tensors of valence 2. $\overset{*}{F}{}^{\alpha\beta}$ is

called the "dual" of $F_{\alpha\beta}$. Also, $\beta = \sqrt{1 - v^2/c^2}$, where v is the velocity of the charge ρ_0.

Four-Dimensional Form of the Equations

In terms of these tensors the conventional four sets assume the form

$$\begin{array}{ll|ll}
\dfrac{\partial H^{\alpha\beta}}{\partial x^\beta} = s^\alpha & \text{I} & \dfrac{\partial s^\alpha}{\partial x^\alpha} = 0 & \text{III} \\[2ex]
\dfrac{\partial \overset{*}{F}{}^{\alpha\beta}}{\partial x^\beta} = 0 & \text{II} & F_{\alpha\beta} = \dfrac{\partial \varphi_\alpha}{\partial x^\beta} - \dfrac{\partial \varphi_\beta}{\partial x^\alpha} & \text{IV}
\end{array} \qquad 14.7$$

For instance, the first set gives

$$\frac{\partial H^{\alpha\beta}}{\partial x^\beta} = \frac{\partial H^{\alpha 1}}{\partial x^1} + \frac{\partial H^{\alpha 2}}{\partial x^2} + \frac{\partial H^{\alpha 3}}{\partial x^3} + \frac{\partial H^{\alpha 4}}{\partial x^4} = s^\alpha$$

when

$$\begin{aligned}
\alpha = 1 \cdots & \frac{\partial H^z}{\partial y} - \frac{\partial H^y}{\partial z} - \frac{1}{c}\frac{\partial D^x}{\partial t} = \frac{\rho_0 V^x}{\beta c} \\
\alpha = 2 \cdots & -\frac{\partial H^z}{\partial x} + \frac{\partial H^x}{\partial z} - \frac{1}{c}\frac{\partial D^y}{\partial t} = \frac{\rho_0 V^y}{\beta c} \\
\alpha = 3 \cdots & \frac{\partial H^y}{\partial x} - \frac{\partial H^x}{\partial y} - \frac{1}{c}\frac{\partial D^z}{\partial t} = \frac{\rho_0 V^z}{\beta c} \\
\alpha = 4 \cdots & \frac{\partial D^x}{\partial x} + \frac{\partial D^y}{\partial y} + \frac{\partial D^z}{\partial z} = \frac{\rho_0}{\beta}
\end{aligned} \qquad 14.8$$

When the velocity of charge is small, v^2/c^2 is negligible compared with unity and $\beta = 1$.

Since $F_{\alpha\beta}$ is skew symmetric, the last set IV only apparently contains $4^2 = 16$ equations. Four of these (when $\alpha = \beta$) are $0 = 0$, while six of the remaining twelve only repeat the other six with a negative sign.

In going over from the *primitive* (rectangular) reference frame hitherto considered to a rectilinear frame, or from a stationary to a *uniformly* moving reference frame, these tensors can be transformed in a routine manner by the formulas previously given, but the conventional forms cannot.

When curvilinear or accelerated reference frames are introduced, these equations have to be generalized again, as will be shown in Chapter XXXI.

EXERCISES

1. Write out the three-dimensional form of equations 14.1–14.4.
2. Write out the four sets of four-dimensional equations 14.7.
3. Let the following C change rectangular axes to cylindrical axes

$$
\mathbf{C} = \begin{array}{c|c|c|c|c|}
 & x^{1'} & x^{2'} & x^{3'} & x^{4'} \\
\hline
x^1 & \cos\theta & -r\sin\theta & & \\
\hline
x^2 & \sin\theta & r\cos\theta & & \\
\hline
x^3 & & & 1 & \\
\hline
x^4 & & & & 1 \\
\hline
\end{array}
\qquad
\begin{array}{c|c|c|c|c|}
 & x^1 & x^2 & x^3 & x^4 \\
\hline
x^\alpha = & x & y & z & jct \\
\hline
\end{array}
\qquad
\begin{array}{c|c|c|c|c|}
 & x^{1'} & x^{2'} & x^{3'} & x^{4'} \\
\hline
x^{\alpha'} = & r & \theta & z & jct \\
\hline
\end{array}
$$

(*a*) Find along the cylindrical axes φ_α, s_α, $F_{\alpha\beta}$, $F^{\alpha\beta}$, and $II^{\alpha\beta}$.
(*b*) Establish Maxwell's equations along the cylindrical axes.

PART II

ROTATING MACHINERY

CHAPTER 15

GENERALIZATION POSTULATES *

A Preliminary Postulate

The purpose of mathematics is to express as long a train of thought as possible with as few symbols as possible.

Suppose, in performing an experiment, it is found that a spring with a spring constant 10 is elongated 2 inches by the application of a force of 20 pounds. That relation is written as $20 = 10 \times 2$. When 30 pounds is applied, the elongation is found to be 3 inches or $30 = 10 \times 3$. For the infinite possible applied forces and for the infinite variety of spring constants a separate equation has to be written.

Algebra introduces the following labor-saving symbolism. Let all the possible displacement be denoted by d, the spring constants by k, and forces by f. Then all possible measurements may be expressed as $f = kd$. That is, it can be postulated that:

An infinite variety of arithmetic equations may be replaced by one algebraic equation of the same form if each number is replaced by an appropriate letter.

Such a replacement shortens the analysis of a problem and offers a better visualization. Nevertheless, at the end of the analysis all letters have to be replaced again by numbers and a certain amount of numerical work performed in spite of the *intermediate* use of algebra.

By long usage this generalization postulate has become second nature to the engineer, and he hardly ever stops to think of it as such.

The First Generalization Postulate

Let a particular network with n meshes be given. For the first mesh an algebraic equation of the form $e_1 = z_1 i_1$ may be written (in conformity with the preliminary postulate); for the second mesh, $e_2 = z_2 i_2$; and so on. Instead of writing n equations and manipulating them, the analysis may be simplified by introducing a new symbolism the following way.

Let all the n mesh currents, i^1, i^2 $\cdots$ be arranged as a 1-matrix and denoted by a single symbol **i**, similarly all the n impressed voltages by **e**.

* *T.A.N.*, Chapters II and III.

Also let all the n^2 self and mutual impedances be arranged as a 2-matrix and denoted by $\mathbf{Z}$. Then the n algebraic equations may be replaced by the single matric equation $\mathbf{e} = \mathbf{Z}\cdot\mathbf{i}$. That is, it can be postulated that:

The n algebraic equations describing a physical system with n degrees of freedom may be replaced by a single equation having the same form as that of a single unit of the system, if each letter is replaced by an appropriate n-matrix. The manipulation of the matric equation follows closely that of the algebraic equation.

Such a replacement shortens the analysis and offers a better visualization than the original n equations. Again, at the end of the analysis:

1. The n-matrices must be replaced by their elements of algebraic letters.
2. The letters must be replaced by numbers.

The Second Generalization Postulate

Instead of one particular network let, say, all the possible stationary networks with n meshes be given. The matric equation of the first network is $\mathbf{e}_1 = \mathbf{Z}_1\cdot\mathbf{i}_1$ (in conformity with the first generalization postulate); that of the second network, $\mathbf{e}_2 = \mathbf{Z}_2\cdot\mathbf{i}_2$; and so on. Instead of analyzing each network separately, it is possible to develop equations that are equally valid for all these networks by introducing the following symbolism.

First let the whole group of all possible transformation matrices $\mathbf{C}_1, \mathbf{C}_2 \cdots = C^{\alpha}_{\alpha'}$ be established (at least, it must be known how to establish them if and when they are needed) that transform any one of the networks into any of the others. If, and only if, these $\mathbf{C}$'s are known, then let the totality of all the current matrices $\mathbf{i}_1, \mathbf{i}_2 \cdots$ be denoted by the contravariant vector (tensor of valence 1) i^{α}, all voltage matrices by the covariant vector e_{α}, and all impedance matrices by the tensor of valence 2, $Z_{\alpha\beta}$. In that case the large number of matric equations may be replaced by the single tensor equation $e_{\alpha} = Z_{\alpha\beta}i^{\beta}$ (or in direct notation $\mathbf{e} = \mathbf{Z}\cdot\mathbf{i}$). That is, it can be postulated that:

If the matric equation of a particular physical system is known, the same equation is valid for a large number of physical systems of the same nature (for which a group of transformation matrices $C^{\alpha}_{\alpha'}$ may be established) if each n-matrix is replaced by an appropriate tensor.

It cannot be sufficiently emphasized that the key to the tensor equation is the existence of the group of transformation matrices $C^{\alpha}_{\alpha'}$ with the aid of which the ordinary equations of any system can be changed at will to those of any other system. It is incorrect to say that a matric equation is valid for, say, all networks. In order that a symbolic equa-

tion, say $\mathbf{Z}' = \mathbf{Z}_1 - \mathbf{Z}_2 \cdot \mathbf{Z}_4^{-1} \cdot \mathbf{Z}_3$, should be valid for *all* networks, *it is absolutely necessary* to know how to establish the components of each of the symbols $\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}_4$, and $\mathbf{Z}'$ for any particular network with the aid of $\mathbf{C}$ from those of any other network by means of definite laws of transformation. But, if $\mathbf{C}$ is known, and $\mathbf{Z}_1$, $\mathbf{Z}_2 \cdots$ each has definite laws of transformations, the latter symbols are not "matrices" but "tensors," actual physical entities.

Once the solution to a problem is expressed in a tensor equation, again, for any particular physical problem:

1. Each tensor must be replaced by the components along the reference frame in question, namely, by an n-matrix.
2. Each n-matrix must be replaced by its algebraic letters.
3. Each letter must be replaced by a number.

Further Generalization Postulates

Since the second postulate refers to *physical systems of the same nature* (or reference frames of the same nature), the question arises what happens if the physical systems are of different nature; say one is a stationary network, the other a rotating machine; or one has a rectilinear, the other a curvilinear, reference frame. For such extensions further generalization postulates can be established that will be covered subsequently.

In general, the greater the saving in thought and labor in the intermediary steps, the more routine work has to be left to be performed at the end of the analysis. In the solution of any problem about the same amount of numerical work has to be performed with or without the use of algebra; the same is true about the use of tensors. Both algebra and tensors are thought-saving and not arithmetic-saving tools. They avoid the necessity of learning a new trick for every problem.

CHAPTER 16

THE PRIMITIVE ROTATING MACHINE *

Reasoning with the Aid of the Generalization Postulates

Let a single coil, in which the instantaneous current i flows, move with an instantaneous velocity $p\theta$ in a magnetic field. A *stationary* observer is able to establish from several numerical experiments (with the aid of the preliminary postulate) algebraic equations for the voltage and torque in the coil. These equations are

$$e = Ri + \frac{d\varphi}{dt} + Bp\theta \qquad 16.1$$

$$f = iB \qquad 16.2$$

where φ is the *flux linkage* of the coil and B is the *flux density* (different from φ) that the coil cuts.

Let a particular rotating machine with *stationary* reference frames be considered, say an amplidyne, in which *several* coils have the same instantaneous velocity $p\theta$. (To simplify the problem, first the equations of *one* machine are developed so that only one $p\theta$ occurs, also only *stationary* reference frames. The extension for several $p\theta$ and for rotating reference frames requires more advanced concepts of tensor analysis.)

By the first generalization postulate, in terms of matrices the above equations assume the form

$$\mathbf{e} = \mathbf{R}\cdot\mathbf{i} + \frac{d\boldsymbol{\varphi}}{dt} + \mathbf{B}p\theta \qquad 16.3$$

$$f = \mathbf{i}\cdot\mathbf{B} \qquad 16.4$$

where $\mathbf{e}$, $\mathbf{i}$, $\boldsymbol{\varphi}$, and $\mathbf{B}$ become 1-matrices and $\mathbf{R}$ becomes a 2-matrix.

By the second generalization postulate, the equations of *all* rotating machines with stationary reference frames become, in terms of tensors,

$$\mathbf{e} = \mathbf{R}\cdot\mathbf{i} + \frac{d\boldsymbol{\varphi}}{dt} + \mathbf{B}p\theta \quad \Big| \quad e_\alpha = R_{\alpha\beta}i^\beta + \frac{d\varphi_\alpha}{dt} + B_\alpha p\theta \qquad 16.5$$

$$f = \mathbf{i}\cdot\mathbf{B} \quad \Big| \quad f = i^\alpha B_\alpha \qquad 16.6$$

* *A.T.E.M.*, Part III, p. 24.

if, and only if, the group of transformation matrices $C^{\alpha}_{\alpha'}$ exists by which the equation of any machine may be established from that of any other.

It should be noted that the reference axes are restricted to be all of the same type, namely, all stationary in space.

The Method of Attack

(*a*) The second postulate suggests that, in order to establish the equations of any machine, first *from the fundamental laws of electrodynamics let the equations of another machine, say the "primitive" machine, be established whose equations are comparatively easy to determine.* Then, by setting up a $C^{\alpha}_{\alpha'}$ between the primitive machine and any other machine, the equations of the latter can be established in a routine manner, without starting its analysis all over again from fundamental laws.

The study of rotating machines (just like the study of general networks) will consist then of three main steps:

1. The establishment of equations of the primitive machine.
2. The establishment of **C** for each machine, showing how the given machine differs from the primitive machine.
3. The *routine* determination of the performance of any machine.

(*b*) Because for special types of machines special labor-saving devices can be introduced, the study of some of these will also be undertaken. All the labor-saving methods for general networks will be used in rotating machines, in addition to new ones. These old devices are:

1. Permanently short-circuited meshes (with or without impressed voltages) are eliminated. This step decreases the number of variables, without, however, changing the degree in $p = d/dt$.

2. Magnetizing currents are eliminated. This step decreases the number of variables, also the degree in $p = d/dt$.

3. Hypothetical design constants (such as "bucking" reactances) are introduced. This step decreases the number of design constants.

4. Hypothetical reference frames (such as "symmetrical components") are used. This step decreases neither the number of variables, nor the degree in p, nor the number of design constants. However, it decreases the number of terms (components of **Z**) and thereby simplifies the inverse calculations.

5. In balanced polyphase machines all but one phase are eliminated. This step decreases greatly the number of variables, the degree in p, the number of design constants, and the number of terms.

Representation of a "Layer of Winding"

The primitive rotating machine consists of a cylindrical stator and a rotor, each equipped with *several* concentric layers of windings. The stator has two salient poles; the rotor is smooth. The simplest element is now not a "coil" but a "layer of winding." For the sake of simplicity, a two-pole, two-phase machine is considered.

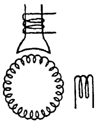


FIG. 16.1. Representation of a stator and rotor layer of winding.

On the *stator* a layer of winding will usually be represented by two coils, one on the salient pole and another at right angles to it between the two poles (Fig. 16.1). On the *rotor* a layer of winding will be represented by a closed drum winding with two sets of brushes on it, one along the field pole (direct axis) and one at right angles to it (quadrature axis). Through the direct axis brush flows i^d; through the quadrature axis brush flows i^q. (A machine with a structure such as Fig. 16.3 is, for instance, the amplidyne.)

Phase-Wound and Squirrel-Cage Rotors

(*a*) D-c. and a-c. commutator machines do have rotor layers of windings equipped with commutators. It can be shown that phase-wound and squirrel-cage rotors also can be represented by a closed drum winding with two *hypothetical* sets of brushes at right angles in space, that serve as reference axes.

If a cross section is made of such a winding, it can be assumed that at

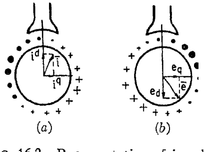


FIG. 16.2. Representation of i and e.

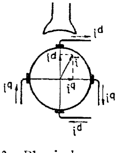


FIG. 16.3. Physical representation of i^d and i^q.

any one instant the *current*-density wave is sinusoidally distributed in *space* (Fig. 16.2*a*). This current will be represented by a vector **i** drawn in the direction of the flux produced by the current. As time goes on, this vector changes its magnitude and direction. The projection of this current (or rather m.m.f.) vector along the salient pole (direct axis) will be denoted by i^d and along the interpolar space (quadrature axis) by i^q.

Similarly an instantaneous generated *voltage* **e** in the winding (Fig. 16.2*b*) will be assumed to be sinusoidal in space and is represented in exactly the same manner as the current **i**.

(*b*) A physical interpretation may be given for i^d and i^q by assuming two hypothetical sets of brushes on the rotor (Fig. 16.3). Then i^d is assumed to flow through the direct axis brushes and i^q through the quadrature axis brushes. Only in commutator machines have these brushes actual physical existence; in synchronous and induction machines they serve only as a reference frame along which the actually existing current vector is projected.

(*c*) To summarize, in the rotor of a commutator machine i^d and i^q each has actual physical existence, but their resultant in space, **i**, is hypothetical. On the other hand, in a phase-wound or a squirrel-cage rotor the resultant **i** has an actual physical existence, and its two components, i^d and i^q, are hypothetical quantities.

The Primitive Machine

A rotor layer of winding with true or hypothetical brushes may be considered to consist of two hypothetical coils *at right angles* (Fig. 16.4).

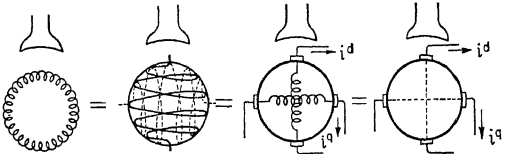


FIG. 16.4. Four different representations of a rotor layer of winding.

While the conductors forming these coils rotate, the resultant coils between the brushes are stationary; that is, the coils are composed of dif-

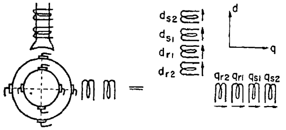
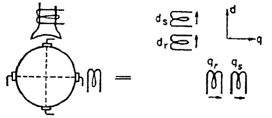


FIG. 16.5. The primitive machine with four layers of windings.

ferent conductors from instant to instant. (In practice the coils are shown by dotted lines, Fig. 16.4*d*.)

Since every layer of winding may be represented by two coils at right angles, the "primitive" machine consists of two sets of coils at right angles in space (Fig. 16.5). To simplify the equations, usually only one



(a) (b)

FIG. 16.6. The primitive machine with two layers of windings.

layer will be assumed on the stator and one on the rotor—four coils altogether (Fig. 16.6). The generalization of all equations from four coils to n coils is obvious.

Generated Voltages

(a) In the primitive machine let a current i^{ds} flow in the stator direct axis winding in the positive direction (producing a positive flux), and let the rotor rotate clockwise. The question to be investigated is: What are the voltages induced and generated in the four windings due to the presence of the single current. The self-inductance of the coil is L_{ds}; its mutual inductance with the rotor is M_d (Fig. 16.7).

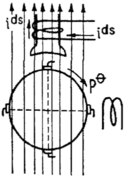


FIG. 16.7. Current i^{ds} flows.

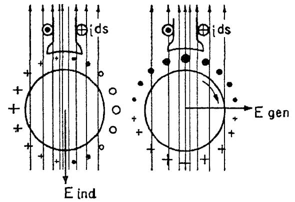


FIG. 16.8. Induced and generated voltage due to i^{ds}.

1. Assuming the rotor stationary and the current varying, voltages are induced only along the direct axis windings $\mathbf{d}_s$ and $\mathbf{d}_r$. In the stator $\mathbf{d}_s$ appears $e = L_{ds}pi^{ds}$, and between the direct axis brushes $\mathbf{d}_r$ appears $e = M_d pi^{ds}$.

2. Assuming the current constant and the rotor rotating with a velocity $p\theta$, generated voltage exists only between the brushes along the

quadrature axis $\mathbf{q}_r$, namely, $e = (M'_d p\theta)i^{ds}$, where M'_d is different from M_d and represents the proportionality factor between the generated voltage e and $i^{ds}\, p\theta$. This factor M'_d will be called here the "mutual inductance" between the axes $\mathbf{d}_s$ and $\mathbf{q}_r$ due to the existence of rotation. The proportionality factor $i^{ds}\, M'_d$ between e and $p\theta$ will be called here the "flux-density wave" B.

In a commutator machine all these induced and generated voltages can be measured and the constants L_{ds}, M_d, and M'_d ascertained by test. In a phase-wound or squirrel-cage motor these constants can also be determined from measurements or design data. In the latter machine the corresponding induced and generated voltages can be represented by Lenz' law as space vectors, as shown in Fig. 16.8.

These internal generated voltages due to i^{ds} (also the resistance drop) may be tabulated as

$$
\begin{aligned}
E_{ds} &= (-r_{ds} - L_{ds}p)i^{ds} \\
E_{dr} &= -M_d p i^{ds} \\
E_{qr} &= M'_d p\theta i^{ds} \\
E_{qs} &= 0
\end{aligned}
\quad = \quad
\begin{array}{c}
i^{ds} \\
\left[\begin{array}{c}
-r_{ds} - L_{ds}p \\ \hline
-M_d p \\ \hline
M'_d p\theta \\ \hline
0
\end{array}\right]
\end{array}
\qquad 16.7
$$

(*b*) If positive currents are assumed to flow in each of the four coils and the voltages due to the presence of each coil current are similarly tabulated, the resultant impedance matrix for the primitive machine becomes

$$
\mathbf{Z}_g = \begin{array}{c|cccc}
 & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\ \hline
\mathbf{d}_s & -r_{ds} - L_{ds}p & -M_d p & 0 & 0 \\
\mathbf{d}_r & -M_d p & -r_r - L_{dr}p & -L'_{qr}p\theta & -M'_q p\theta \\
\mathbf{q}_r & M'_d p\theta & L'_{dr}p\theta & -r_r - L_{qr}p & -M_q p \\
\mathbf{q}_s & 0 & 0 & -M_q p & -r_{qs} - L_{qs}p
\end{array}
\qquad
\mathbf{e}_g = \begin{array}{c|c}
\mathbf{d}_s & E_{ds} \\
\mathbf{d}_r & E_{dr} \\
\mathbf{q}_r & E_{qr} \\
\mathbf{q}_s & E_{qs}
\end{array}
\qquad 16.8
$$

so that the *generated* voltage equation is $\mathbf{e}_g = \mathbf{Z}_g \cdot \mathbf{i}$.

That is, the $\mathbf{Z}_g$ of the primitive machine is the same as the $\mathbf{Z}_g$ of a d-c. machine with two sets of brushes at right angles.

In establishing $\mathbf{Z}_g$ it has been assumed that each coil has the same number of turns. The inductances L and r may be measured in henries or in any per unit system. The instantaneous velocity $p\theta$ represents in general the number of electrical radians described per second.

These basic equations are used by central-station engineers (Park, Crary, Concordia, and others) for the study of synchronous machines.

(c) Induction-motor engineers usually know the impressed voltages $\mathbf{e}$ on the motor; hence they prefer to use the negative of the above equations

$$-\mathbf{e}_g = -\mathbf{Z}_g \cdot \mathbf{i} \quad \text{or} \quad \mathbf{e} = \mathbf{Z} \cdot \mathbf{i} \qquad 16.9$$

where $\mathbf{Z}$ is the negative of $\mathbf{Z}_g$.

$$\mathbf{Z} = \begin{array}{c|cccc} & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\ \hline \mathbf{d}_s & r_{ds} + L_{ds}p & M_d p & 0 & 0 \\ \mathbf{d}_r & M_d p & r_r + L_{dr}p & L'_{qr}p\theta & M'_q p\theta \\ \mathbf{q}_r & -M'_d p\theta & -L'_{dr}p\theta & r_r + L_{qr}p & M_q p \\ \mathbf{q}_s & 0 & 0 & M_q p & r_{qs} + L_{qs}p \end{array} \qquad 16.10$$

(In machines with smooth air gaps $L_{dr} = L_{qr} = L_r$ and $M_d = M_q = M$.) Also

$$\mathbf{e} = \begin{array}{cccc} \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\ \hline e_{ds} & e_{dr} & e_{qr} & e_{qs} \end{array} = -\mathbf{e}_g = \begin{array}{cccc} \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\ \hline -E_{ds} & -E_{dr} & -E_{qr} & -E_{qs} \end{array} \qquad 16.11$$

where the symbols e represent *impressed* voltages and

$$\mathbf{i} = \begin{array}{cccc} \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\ \hline i^{ds} & i^{dr} & i^{qr} & i^{qs} \end{array}$$

Hence the impressed-voltage equations of the primitive machine are

$$\begin{aligned} e_{ds} &= (r_{ds}+L_{ds}p)i^{ds} + M_d p\, i^{dr} \\ e_{dr} &= M_d p\, i^{ds} + (r_r + L_{dr}p)i^{dr} + L'_{qr}p\theta\, i^{qr} + M'_q p\theta\, i^{qs} \\ e_{qr} &= -M'_d p\theta\, i^{ds} - L'_{dr}p\theta\, i^{dr} + (r_r + L_{qr}p)i^{qr} + M_q p\, i^{qs} \\ e_{qs} &= M_q p\, i^{qr} + (r_{qs}+L_{qs}p)i^{qs} \end{aligned} \qquad 16.12$$

(d) As the order in which the axes are considered is arbitrary, *any other order may be assumed at will.* For instance, in using symmetrical

components, sometimes it is more convenient to assume the order $\mathbf{d}_s$, $\mathbf{q}_s$, $\mathbf{d}_r$, $\mathbf{q}_r$ so that

$$\mathbf{Z} = \begin{array}{c|c|c|c|c|} & \mathbf{d}_s & \mathbf{q}_s & \mathbf{d}_r & \mathbf{q}_r \\ \hline \mathbf{d}_s & r_{ds} + L_{ds}p & 0 & M_d p & 0 \\ \hline \mathbf{q}_s & 0 & r_{qs} + L_{qs}p & 0 & M_q p \\ \hline \mathbf{d}_r & M_d p & M_q' p\theta & r_r + L_{dr}p & L_{qr}' p\theta \\ \hline \mathbf{q}_r & -M_d' p\theta & M_q p & -L_{dr}' p\theta & r_r + L_{qr}p \\ \hline \end{array} \qquad 16.13$$

Component Tensors of Z

(*a*) The above impedance tensor consists of the sum of three tensors:

1. The coefficients of all p are denoted by **L**.
2. The coefficients of all $p\theta$ are denoted by **G**.
3. The remaining terms are denoted by **R**.

$$\mathbf{R} = \begin{array}{c|c|c|c|c|} & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\ \hline \mathbf{d}_s & r_{ds} & & & \\ \hline \mathbf{d}_r & & r_r & & \\ \hline \mathbf{q}_r & & & r_r & \\ \hline \mathbf{q}_s & & & & r_{qs} \\ \hline \end{array} \quad \mathbf{L} = \begin{array}{c|c|c|c|c|} & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\ \hline \mathbf{d}_s & L_{ds} & M_d & & \\ \hline \mathbf{d}_r & M_d & L_{dr} & & \\ \hline \mathbf{q}_r & & & L_{qr} & M_q \\ \hline \mathbf{q}_s & & & M_q & L_{qs} \\ \hline \end{array} \quad \mathbf{G} = \begin{array}{c|c|c|c|c|} & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\ \hline \mathbf{d}_s & & & & \\ \hline \mathbf{d}_r & & & L_{qr}' & M_q' \\ \hline \mathbf{q}_r & -M_d' & -L_{dr}' & & \\ \hline \mathbf{q}_s & & & & \\ \hline \end{array}$$

16.14

1. The resistance tensor **R** contains the resistances of the four windings.

2. The inductance tensor **L** contains the self and mutual inductances of the four windings. (There is no mutual inductance between the direct and quadrature windings.) The inductance tensor $\mathbf{L} = L_{\alpha\beta}$ plays a fundamental role in tensor analysis and is called the "metric tensor." In dynamical studies the metric tensor is denoted by $a_{\alpha\beta}$ and in geometry by $g_{\alpha\beta}$.

3. The torque tensor **G** contains the mutual inductances existing because of *rotation* (such mutuals exist only between **d** and **q** coils).

(*b*) In terms of the three tensors

$$\mathbf{Z} = \mathbf{R} + \mathbf{L}p + p\theta\mathbf{G} \ \Big|\ Z_{\alpha\beta} = R_{\alpha\beta} + L_{\alpha\beta}p + p\theta G_{\alpha\beta} \qquad 16.15$$

so that $\mathbf{e} = \mathbf{Z}\cdot\mathbf{i}$ may be written as

$$\mathbf{e} = \mathbf{R}\cdot\mathbf{i} + \mathbf{L}\cdot p\mathbf{i} + p\theta\mathbf{G}\cdot\mathbf{i} \mid e_\alpha = R_{\alpha\beta}i^\beta + L_{\alpha\beta}pi^\beta + p\theta G_{\alpha\beta}i^\beta \qquad 16.16$$

Physical Tensors

(*a*) From the basic tensors **R, L,** and **G** (containing design constants), other tensors may be derived expressing physical entities. Two of these tensors are:

1. The flux-linkage vector φ representing the *resultant* flux linkages of each winding

$$\varphi = \mathbf{L}\cdot\mathbf{i} \mid \varphi_\alpha = L_{\alpha\beta}i^\beta \qquad 16.17$$

2. The flux-density vector **B** representing the *resultant* flux density cut by each coil

$$\mathbf{B} = \mathbf{G}\cdot\mathbf{i} \mid B_\alpha = G_{\alpha\beta}i^\beta \qquad 16.18$$

In terms of these vectors

$$\mathbf{e} = \mathbf{R}\cdot\mathbf{i} + p\varphi + \mathbf{B}p\theta \mid e_\alpha = R_{\alpha\beta}i^\beta + p\varphi_\alpha + B_\alpha p\theta \qquad 16.19$$

(*b*) For the primitive machine

$$\varphi = \begin{array}{c|c|} \mathrm{d}_s & L_{ds}i^{ds} + M_d i^{dr} \\ \mathrm{d}_r & M_d i^{ds} + L_{dr}i^{dr} \\ \mathrm{q}_r & L_{qr}i^{qr} + M_q i^{qs} \\ \mathrm{q}_s & M_q i^{qr} + L_{qs}i^{qs} \end{array} \qquad \mathbf{B} = \begin{array}{c|c|} \mathrm{d}_s & 0 \\ \mathrm{d}_r & L'_{qr}i^{qr} + M'_q i^{qs} \\ \mathrm{q}_r & -(M'_d i^{ds} + L'_{dr}i^{dr}) \\ \mathrm{q}_s & 0 \end{array} \qquad 16.20$$

The flux-density vector **B** represents only the *rotor* flux densities that produce generated voltages and torques. The stator flux densities play no role in these equations.

In terms of φ and **B** the equations 16.19 of the primitive machine are

$$\begin{aligned} e_{ds} &= r_{ds}i^{ds} + p\varphi_{ds} \\ e_{dr} &= r_r i^{dr} + p\varphi_{dr} + B_{dr}p\theta \\ e_{qr} &= r_r i^{qr} + p\varphi_{qr} + B_{qr}p\theta \\ e_{qs} &= r_{qs}i^{qs} + p\varphi_{qs} \end{aligned} \qquad 16.21$$

(*c*) Since the electromagnetic torque upon the rotor (in the direction θ) is

$$f = \mathbf{i}\cdot\mathbf{B} \mid f = i^\alpha B_\alpha \qquad 16.22$$

substituting the value of **B**, the *instantaneous torque* is

$$f = \mathbf{i}\cdot\mathbf{G}\cdot\mathbf{i} \;\Big|\; f = G_{\alpha\beta}i^{\alpha}i^{\beta} \qquad 16.23$$

The Rotation Tensor

(*a*) In phase-wound and squirrel-cage rotors it is assumed that the current-density and flux-density waves are sinusoidally distributed in space. In that case $M' = M$ and $L' = L$. Also **G** may be expressed in terms of **L** as

$$\mathbf{G} = \boldsymbol{\gamma}_t\cdot\mathbf{L} \qquad G_{\alpha\beta} = \gamma_{\alpha}^{\gamma}L_{\gamma\beta} \qquad 16.24$$

where

$$\boldsymbol{\gamma}_t = \gamma_{\alpha}^{\gamma} = \begin{array}{c|cccc} {}_{\alpha}\diagdown^{\gamma} & d_s & d_r & q_r & q_s \\ \hline d_s & & & & \\ d_r & & & 1 & \\ q_r & & -1 & & \\ q_s & & & & \end{array} \qquad 16.25$$

A similar relation exists between the flux-density wave **B** and flux-linkage wave φ

$$\mathbf{B} = \boldsymbol{\gamma}_t\cdot\varphi \;\Big|\; B_{\alpha} = \gamma_{\alpha}^{\beta}\varphi_{\beta} \qquad 16.26$$

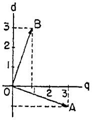


FIG. 16.9. Rotating a vector by 90° with the aid of $\boldsymbol{\gamma}_t$.

(*b*) The tensor $\boldsymbol{\gamma}$ is called the "rotation tensor" as it rotates a vector in space by 90 degrees. For instance, if in Fig. 16.9

$$\mathbf{i} = \begin{array}{cc} d & q \\ \hline -1 & 3 \end{array} = OA$$

then

$$\boldsymbol{\gamma}_t\cdot\mathbf{i} = \begin{array}{cc} d & q \\ \hline 3 & 1 \end{array} = OB$$

Hence in each layer of winding the flux-density wave **B** is at right angles in space from the flux-linkage wave φ.

(*c*) *In a commutator machine* γ_t *has no existence,* G *has no relation to* L, *and* B *is independent of* φ.

More General Forms of Z

(*a*) When the rotor rotates in the opposite direction (counterclockwise), Fig. 16.10, then $p\theta$ becomes negative and

$$
\mathbf{Z} = \begin{array}{c|c|c|c|c|}
 & \mathrm{d}_s & \mathrm{d}_r & \mathrm{q}_r & \mathrm{q}_s \\ \hline
\mathrm{d}_s & r_{ds} + L_{ds}p & M_d p & & \\ \hline
\mathrm{d}_r & M_d p & r_r + L_{dr}p & -L'_{qr}p\theta & -M'_q p\theta \\ \hline
\mathrm{q}_r & M'_d p\theta & L'_{dr}p\theta & r_r + L_{qr}p & M_q p \\ \hline
\mathrm{q}_s & & & & r_{qs} + L_{qs}p \\ \hline
\end{array} \qquad 16.27
$$

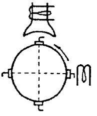


FIG. 16.10. Primitive machine of Z.

The equation of voltage becomes

$$\mathbf{e} = \mathbf{Z}\cdot\mathbf{i} = \mathbf{R}\cdot\mathbf{i} + p\varphi - \mathbf{B}p\theta \;\Big|\; e_\alpha = Z_{\alpha\beta}i^\beta = R_{\alpha\beta}i^\beta + p\varphi_\alpha - B_\alpha p\theta \qquad 16.28$$

(*b*) When zero-phase sequence currents flow in the stator layer, or rotor layer, or both, *an extra row and column are introduced in* Z *for each zero-sequence current* and an extra coil in the primitive machine. Then

$$
\mathbf{Z} = \begin{array}{c|c|c|c|c|c|c|}
 & \mathrm{d}_s & \mathrm{d}_r & \mathrm{q}_r & \mathrm{q}_s & 0_s & 0_r \\ \hline
\mathrm{d}_s & r_{ds} + L_{ds}p & M_d p & & & & \\ \hline
\mathrm{d}_r & M_d p & r_r + L_{dr}p & L'_{qr}p\theta & M'_q p\theta & & \\ \hline
\mathrm{q}_r & -M'_d p\theta & -L'_{dr}p\theta & r_r + L_{qr}p & M_q p & & \\ \hline
\mathrm{q}_s & & & M_q p & r_{qs} + L_{qs}p & & \\ \hline
0_s & & & & & r_{s0} + L_{s0}p & \\ \hline
0_r & & & & & & r_{r0} + L_{r0}p \\ \hline
\end{array} \qquad 16.29
$$

That is, now *three axes exist on each layer of winding,* d, q, *and* 0.

(30)

	$\mathbf{d}_{s2}$	$\mathbf{d}_{s1}$	$\mathbf{d}_{r1}$	$\mathbf{d}_{r2}$	$\mathbf{q}_{r2}$	$\mathbf{q}_{r1}$	$\mathbf{q}_{s1}$	$\mathbf{q}_{s2}$
	$r_{ds2} + L_{ds2}p$	$M_{ds}p$	$M_{d12}p$	$M_{d22}p$	0	0	0	0
	$M_{ds}p$	$r_{ds1} + L_{ds1}p$	$M_{d11}p$	$M_{d21}p$	0	0	0	0
	$M_{d12}p$	$M_{d11}p$	$r_{r1} + L_{dr1}p$	$M_{dr}p$	$M'_{qr}p\theta$	$L'_{qr1}p\theta$	$M'_{q11}p\theta$	$M'_{q12}p\theta$
	$M_{d22}p$	$M_{d21}p$	$M_{dr}p$	$r_{r2} + L_{dr2}p$	$L'_{qr2}p\theta$	$M'_{qr}p\theta$	$M'_{q21}p\theta$	$M'_{q22}p\theta$
	$-M'_{d22}p\theta$	$-M'_{d21}p\theta$	$-M'_{dr}p\theta$	$-L'_{dr2}p\theta$	$r_{r2} + L_{qr2}p$	$M_{qr}p$	$M_{q21}p$	$M_{q22}p$
	$-M'_{d12}p\theta$	$-M'_{d11}p\theta$	$-L'_{dr}p\theta$	$-M'_{dr}p\theta$	$M_{qr}p$	$r_{r1} + L_{qr1}p$	$M_{q11}p$	$M_{q12}p$
	0	0	0	0	$M_{q21}p$	$M_{q11}p$	$r_{qs1} + L_{qs1}p$	$M_{qs}p$
	0	0	0	0	$M_{q22}p$	$M_{q12}p$	$M_{qs}p$	$r_{qs2} + L_{qs2}p$

(*c*) With two layers of windings on the stator and rotor, Fig. 16.5 (but no zero-sequence currents), the **Z** is shown in equation 16.30. All components containing p represent **L**; those containing $p\theta$ represent **G**; and the rest, **R**.

Axes Fixed to the Rotor

(*a*) It is a property of the laws of electrodynamics that they depend only on the *relative* velocities existing between the reference frames, the electromagnetic field, and the material bodies lying in the field.

In the primitive machine of Fig. 16.6 (having the Z of equation 16.10):

1. The four reference axes and the salient pole are stationary.
2. The smooth structure rotates clockwise.

Hence the same **Z** is valid also if:

1. The smooth structure is stationary,
2. The four reference axes and the salient pole rotate together counterclockwise (Fig. 16.11).

(*b*) Such a case occurs in synchronous machines (Fig. 16.12); hence the **Z** of equations 16.10 and 16.30 are equally valid for them, if the sub-

FIG. 16.11. Relative rotations of a salient and smooth structures.

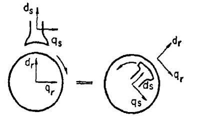


FIG. 16.12. Equivalence of induction machine and synchronous machine structures.

script s refers to the salient pole (now the rotor) and the subscript r to the armature (now stationary).

However, synchronous-motor engineers assume that the salient pole (and the reference frame) rotates *clockwise* (or rather from **d** to **q**); hence it is the Z of equation 16.27 that corresponds to this convention. Since usually amortisseur windings (axes **k**) exist in both direct and quadrature axes, *a primitive machine with at least five axes appears in synchronous-machine studies*. Hence extending equation 16.27 in the manner of equation 16.30, and replacing the subscripts s by f (field) and k (amortisseur) also r by a (armature), the **Z** to be used in synchronous machine studies is

$$
\begin{array}{c|c|c|c|c|c|}
 & \mathbf{d}_f & \mathbf{d}_k & \mathbf{d}_a & \mathbf{q}_a & \mathbf{q}_k \\
\hline
\mathbf{d}_f & -r_f - L_f p & -M_{fk} p & -M_{fd} p & & \\
\hline
\mathbf{d}_k & -M_{fk} p & -r_{kd} - L_{kd} p & -M_{kd} p & & \\
\hline
\mathbf{d}_a & -M_{fd} p & -M_{kd} p & -r - L_d p & L_q p\theta & M_{kq} p\theta \\
\hline
 & -M_{fd} p\theta & -M_{kd} p\theta & -L_d p\theta & -r - L_q p & -M_{kq} p \\
\hline
\mathbf{q}_k & & & & -M_{kq} p & -r_{kq} - L_{kq} p \\
\hline
\end{array}
\qquad 16.31
$$

If G is formed (Table V) from the coefficients of $p\theta$ in $\mathbf{Z}_g$ (equation 16.31), then $\mathbf{i}\cdot\mathbf{G}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{B}$ represents the electromagnetic torque on the *stator* in the direction θ since $\mathbf{B}$ now represents the flux-density wave of the stator.

It may be mentioned that *the direction of* q *may be reversed* as shown in Fig. 16.13*c*, *and a counterclockwise rotation (still from* **d** *to* **q**) *may be assumed.* All equations, however, remain the same with both conventions.

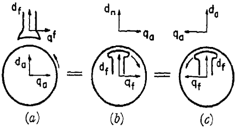


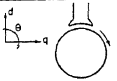
FIG. 16.13. Three different representations of a synchronous machine.

(*c*) When several machines are interconnected, some rotating clockwise, some counterclockwise, then appropriate $\mathbf{Z}$ (or $\mathbf{Z}_g$) has to be used for each. For future reference, Table V has been constructed for $\mathbf{Z}$ and $\mathbf{Z}_g$ containing two directions of rotation. For each type of machine and for each direction of rotation $\mathbf{Z} = -\mathbf{Z}_g$.

(*d*) Even though the Z of a synchronous machine refers to a reference frame that rotates, inasmuch as the four axes are *relatively stationary* with respect to each other both cases will be called "stationary" axes, meaning

TABLE V

THE Z, Zg, AND G TENSORS OF THE PRIMITIVE MACHINE WITH VARIOUS DIRECTIONS OF ROTATION.



a

$Z =$

	d_s	d_r	q_r	q_s
d_s	$r_{ds}+L_{ds}p$	$M_d p$	0	0
d_r	$M_d p$	$r_r+L_{dr}p$	$L'_{qr}p\theta$	$M'_q p\theta$
q_r	$-M'_d p\theta$	$-L'_{dr}p\theta$	$r_r+L_{qr}p$	$M_q p$
q_s	0	0	$M_q p$	$r_{qs}+L_{qs}p$

b

$Z_g =$

	d_s	d_r	q_r	q_s
d_s	$-r_{ds}-L_{ds}p$	$-M_d p$	0	0
d_r	$-M_d p$	$-r_r-L_{dr}p$	$-L'_{qr}p\theta$	$-M'_q p\theta$
q_r	$M'_d p\theta$	$L'_{dr}p\theta$	$-r_r-L_{qr}p$	$-M_q p$
q_s	0	0	$-M_q p$	$-r_{qs}-L_{qs}p$

c

$Z =$

	d_r	d_s	q_s	q_r
d_r	$r_{dr}+L_{dr}p$	$M_d p$	0	0
d_s	$M_d p$	$r_s+L_{ds}p$	$-L'_{qs}p\theta$	$-M'_q p\theta$
q_s	$M'_d p\theta$	$L'_{ds}p\theta$	$r_s+L_{qs}p$	$M_q p$
q_r	0	0	$M_q p$	$r_{qr}+L_{qr}p$

d

$Z_g =$

	d_r	d_s	q_s	q_r
d_r	$-r_{dr}-L_{dr}p$	$-M_d p$	0	0
d_s	$-M_d p$	$-r_s-L_{ds}p$	$L'_{qs}p\theta$	$M'_q p\theta$
q_s	$-M'_d p\theta$	$-L'_{ds}p\theta$	$-r_s-L_{qs}p$	$-M_q p$
q_r	0	0	$-M_q p$	$-r_{qr}-L_{qr}p$

e

$Z =$

	d_s	d_r	q_r	q_s
d_s	$r_{ds}+L_{ds}p$	$M_d p$	0	0
d_r	$M_d p$	$r_r+L_{dr}p$	$-L'_{qr}p\theta$	$-M'_q p\theta$
q_r	$M'_d p\theta$	$L'_{dr}p\theta$	$r_r+L_{qr}p$	$M_q p$
q_s	0	0	$M_q p$	$r_{qs}+L_{qs}p$

$G =$

	d_s	d_r	q_r	q_s
d_s				
d_r			L'_{qr}	M'_q
q_r	$-M'_d$	$-L'_{dr}$		
q_s				

$G =$

	d_r	d_s	q_s	q_r
d_r				
d_s			$-L'_{qs}$	$-M'_q$
q_s	M'_d	L'_{ds}		
q_r				

f

The G tensor can also be divided into two components $\mathbf{G}_s$ and $\mathbf{G}_r$

$$\mathbf{G} = \mathbf{G}_s + \mathbf{G}_r \tag{16.33}$$

$$\mathbf{G} = \begin{array}{c} \\ d_s \\ d_r \\ q_r \\ q_s \end{array}\!\!\begin{array}{cccc} d_s & d_r & q_r & q_s \\ \hline & & & \\ & & L_{qr} & M_q \\ -M_d & -L_{dr} & & \\ & & & \end{array} = \begin{array}{c} \\ \\ d_r \\ q_r \\ \\ \end{array}\!\!\begin{array}{cccc} d_s & & & q_s \\ \hline & & & \\ & & & M_q \\ -M_d & & & \\ & & & \end{array}$$

$$+ \begin{array}{c} \\ \\ d_r \\ q_r \\ \\ \end{array}\!\!\begin{array}{cccc} & d_r & q_r & \\ \hline & & & \\ & & L_{qr} & \\ & -L_{dr} & & \\ & & & \end{array} \tag{16.34}$$

so that

$$f = \mathbf{i}_r \cdot \mathbf{G}_s \cdot \mathbf{i}_s + \mathbf{i}_r \cdot \mathbf{G}_r \cdot \mathbf{i}_r \tag{16.35}$$

(*b*) Now, if the machine is smooth, $L_{qr} = L_{dr} = L_r$, and the *torque due to the rotor currents alone is zero.*

$$\mathbf{i}_r \cdot \mathbf{G}_r \cdot \mathbf{i}_r = i^{dr} L_r i^{qr} - i^{qr} L_r i^{dr} = 0 \tag{16.36}$$

and the torque becomes

$$f = \mathbf{i}_r \cdot \mathbf{G}_s \cdot \mathbf{i}_s \tag{16.37}$$

In machines with salient poles, the torque

$$f = \mathbf{i}_r \cdot \mathbf{G}_r \cdot \mathbf{i}_r = i^{dr} i^{qr} (L_{qr} - L_{dr}) \tag{16.38}$$

is the so-called reaction torque introduced by the saliency of the poles.

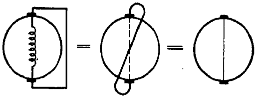
(It must be remembered, that $\mathbf{G}_s$ and $\mathbf{G}_r$ are no longer tensors and they cannot be introduced if, for instance, the equations are intended to be used for establishing equivalent circuits.)

CHAPTER 17

TRANSFORMATION TENSOR*

Interconnecting Coils

(*a*) The primitive machine consists of several isolated coils, each with an e.m.f. (mostly of zero value) impressed upon it. It differs from the primitive stationary network only in one respect. Its coils are arranged at right angles in space, thereby having mutual inductances only between coils along the same axis (as if it consisted of two isolated multi-winding transformers). Because of the permanent spatial arrangement, it is not necessary to denote the ends of the coils by 1–2.



(*a*) (*b*) (*c*)

FIG. 17.1. Representations of a short-circuited brush set.

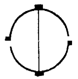


FIG. 17.2. Representation of a squirrel-cage winding.

If the stator and rotor coils of one or more primitive machines are interconnected in any manner with each other or with some stationary network, the steps in establishing **C** are exactly the same as in stationary networks.

(*b*) A set of brushes short-circuited upon itself is represented by a heavy line, Fig. 17.1.

A squirrel-cage winding is represented by two sets of short-circuited brushes at right angles (Fig. 17.2).

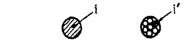
The Turn-Ratio Transformation C

(*a*) If the constants of the primitive machine are calculated by assuming that all coils have the same number of turns, then when two coils are connected in series, their turn ratio must be considered.

* *A.T.E.M.*, Part III.

Let the current in a conductor be i (Fig. 17.3). If the conductor is subdivided into n small but equal conductors, the current in each is i', so that the relation

$$i = ni' \qquad 17.1$$



Before subdivision. After subdivision.

FIG. 17.3. Changing the number of turns.

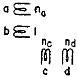


FIG. 17.4. Coils with different number of turns.

represents the transformation of increasing the number of turns of a coil by n. When in the primitive network of Fig. 17.4 coil b has unit turns, while the others have a different number, then

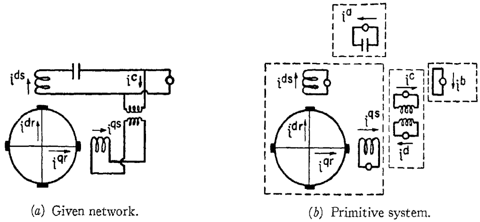
$$i^a = n_a i^{a'}$$
$$i^b = i^{b'}$$
$$i^c = n_c i^{c'}$$
$$i^d = n_d i^{d'}$$

$\mathbf{C}_1 =$

	a'	b'	c'	d'
a	n_a			
b		1		
c			n_c	
d				n_d

17.2

If the coils are now interconnected by $\mathbf{C}_2$, then $\mathbf{C} = \mathbf{C}_1 \cdot \mathbf{C}_2$.



(a) Given network. (b) Primitive system.

FIG. 17.5. Interconnection of a rotating machine with a stationary network.

Because of the simplicity of $\mathbf{C}_1$, it is usually possible to set up $\mathbf{C}_1 \cdot \mathbf{C}_2$ in one step. When in doubt, $\mathbf{C}$ should be set up in two steps.

(b) For instance, let a motor be interconnected with a stationary network as shown in Fig. 17.5a.

The primitive network has eight currents (one for the impressed voltage that happens to have zero impedance in series with it). The given network has five currents. Equating the old and the new currents flowing in each coil (assuming coils $\mathbf{d}_s$, $\mathbf{d}_r$, $\mathbf{q}_r$, and $\mathbf{a}$ to have unit turns)

$$
\begin{aligned}
i^{ds} &= i^{ds'} \\
i^{dr} &= i^{dr'} \\
i^{qr} &= i^{qr'} \\
i^{qs} &= n_q i^{qs'} \\
i^{a} &= i^{ds'} \\
i^{b} &= i^{ds'} - i^{c'} \\
i^{c} &= n_c i^{c'} \\
i^{d} &= -n_d i^{qs'}
\end{aligned}
\qquad
\mathbf{C} = \begin{array}{c|ccccc|}
 & \mathbf{d}_s' & \mathbf{d}_r' & \mathbf{q}_r' & \mathbf{q}_s' & \mathbf{c}' \\
\hline
\mathbf{d}_s & 1 & & & & \\
\mathbf{d}_r & & 1 & & & \\
\mathbf{q}_r & & & 1 & & \\
\mathbf{q}_s & & & & n_q & \\
\mathbf{a} & 1 & & & & \\
\mathbf{b} & 1 & & & & -1 \\
\mathbf{c} & & & & & n_c \\
\mathbf{d} & & & & -n_d & \\
\hline
\end{array}
\qquad 17.3
$$

In many rotating-machine problems the primitive system is so obvious that it is not necessary to make a special drawing such as Fig. 17.5*b*.

Rotation of the Rotor Reference Frame

(*a*) There is one procedure that is performed with the coils of the primitive machine, but not performed with the coils of the primitive stationary network, and that is the rotation of the coils in space, or rather the rotation of the brushes in space. (In stationary networks the spatial position of the coils was not considered.)

The following analysis is valid in commutator machines only approximately, as the current-density and flux-density waves are assumed to be either sinusoidal or at least replaceable by a sinusoidal wave for each position of the brush set. In the latter case all angles are not true, but equivalent angles.

(*b*) Let a cross section of a rotor layer of winding be taken, Fig. 17.6*a*, and let it be assumed that at a certain instant the current vector $\mathbf{i}$ is at the position shown. If the machine is the primitive machine, $\mathbf{i}$ is projected along the $\mathbf{d}$ and $\mathbf{q}$ axes to give i^d and i^q (Fig. 17.6*b*).

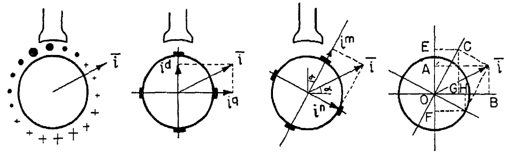
In many actual machines the two sets of brushes $\mathbf{m}$ and $\mathbf{n}$ are at a constant angle α from $\mathbf{d}$ and $\mathbf{q}$; hence in them $\mathbf{i}$ is projected along $\mathbf{m}$ and $\mathbf{n}$ as i^m and i^n. That is (Fig. 17.6*c*),

1. The old projections of $\mathbf{i}$ are i^d and i^q.
2. The new projections of $\mathbf{i}$ are i^m and i^n.

(*c*) The problem is to express the old components i^d and i^q in terms of the new components i^m and i^n.

From Fig. 17.6*d* it is evident that

$$
\begin{array}{l|l|}
OA = OE - OF & i^d = i^m \cos\alpha - i^n \sin\alpha \\
OB = OG + OH & i^q = i^m \sin\alpha + i^n \cos\alpha
\end{array}
\qquad
\mathbf{C} = \begin{array}{c} \\ \mathbf{d} \\ \mathbf{q} \end{array}
\begin{array}{cc} \mathbf{m} & \mathbf{n} \\ \hline \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{array}
\qquad 17.4
$$



(*a*) Current vector. (*b*) Old projections. (*c*) New projections. (*d*) Old projections in terms of new.

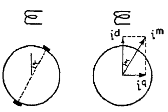
FIG. 17.6. Projecting the current vector **i** upon two sets of reference frames at an angle α.

The coefficients of the new currents give **C** that changes the current components from **d** and **q** to **m** and **n** but leaves the current vector **i** itself invariant (unchanged).

Special Cases

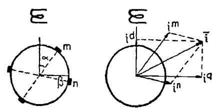
With one set of brushes **m** on the rotor (Fig. 17.7) $i^n = 0$ and

$$
\begin{aligned}
i^d &= i^m \cos\alpha \\
i^q &= i^m \sin\alpha
\end{aligned}
\qquad
\mathbf{C} = \begin{array}{c} \\ \mathbf{d} \\ \mathbf{q} \end{array}
\begin{array}{c} \mathbf{m} \\ \hline \cos\alpha \\ \sin\alpha \end{array}
\qquad 17.5
$$



(*a*) (*b*)

FIG. 17.7. One set of brushes $\mathbf{i} = i^m$.



(*a*) (*b*)

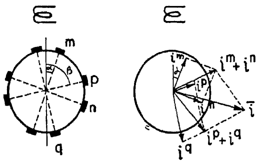
FIG. 17.8. Brushes shifted at different angles, $\mathbf{i} = i^m + i^n$.

When one of the sets of brushes is shifted by an angle α, the other by an angle β (Fig. 17.8), then

$$i^d = i^m \cos\alpha - i^n \sin\beta \qquad C = \begin{array}{c|c|c|} & m & n \\ \hline d & \cos\alpha & -\sin\beta \\ \hline q & \sin\alpha & \cos\beta \\ \hline \end{array} \qquad 17.6$$
$$i^q = i^m \sin\alpha + i^n \cos\beta$$

When the angle of the set of brush (the reference axis) is not a constant α but a function of time θ, the C is the same as above except that α is replaced by θ.

With *four* sets of brushes on a layer of winding (Fig. 17.9) it will be



(*a*) (*b*)

FIG. 17.9. Four sets of brushes on a layer $\mathbf{i} = (i^m + i^n) + (i^p + i^q)$.

assumed that the resultant current **i** is the sum of the currents flowing through the four sets. Hence

$$C = \begin{array}{c|c|c|c|c|} & m & n & p & q \\ \hline d & \cos\alpha & -\sin\alpha & \cos\beta & -\sin\beta \\ \hline q & \sin\alpha & \cos\alpha & \sin\beta & \cos\beta \\ \hline \end{array} \qquad 17.7$$

Establishing C in Several Steps

When the brushes are rotated and interconnected with other coils, it is better to perform the transformation in two steps. First, the brushes

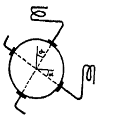


FIG. 17.10. Leblanc advancer.

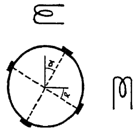


FIG. 17.11. Shifting the brushes by an angle α.

are rotated by C_1, then interconnected by C_2. The product $C_1 \cdot C_2$ performs both operations at the same time. With other complications

(such as different turn ratios) additional **C**'s may be established. For instance, let **C** for the Leblanc advancer (Fig. 17.10) be developed in three steps:

1. Changing the turn ratios.

$$i^{ds} = a i'^{ds} \qquad i^{dr} = i'^{dr} \qquad i^{qr} = i'^{qr} \qquad i^{qs} = a i'^{qs}$$

$$\mathbf{C}_1 = \begin{array}{c|cccc} & \mathrm{d}_s & \mathrm{d}_r & \mathrm{q}_r & \mathrm{q}_s \\ \hline \mathrm{d}_s & a & & & \\ \mathrm{d}_r & & 1 & & \\ \mathrm{q}_r & & & 1 & \\ \mathrm{q}_s & & & & a \end{array} \qquad 17.8$$

2. Shifting the brushes (Fig. 17.11).

$$i^{ds} = i'^{ds} \qquad i^{dr} = i^{m} \cos\alpha - i^{n} \sin\alpha \qquad i^{qr} = i^{m} \sin\alpha + i^{n} \cos\alpha \qquad i^{qs} = i'^{qs}$$

$$\mathbf{C}_2 = \begin{array}{c|cccc} & \mathrm{d}_s & \mathrm{m} & \mathrm{n} & \mathrm{q}_s \\ \hline \mathrm{d}_s & 1 & & & \\ \mathrm{d}_r & & \cos\alpha & -\sin\alpha & \\ \mathrm{q}_r & & \sin\alpha & \cos\alpha & \\ \mathrm{q}_s & & & & 1 \end{array} \qquad 17.9$$

3. Interconnecting coils.

$$i^{ds} = i'^{ds} \qquad i^{m} = i'^{ds} \qquad i^{n} = i'^{qs} \qquad i^{qs} = i'^{qs}$$

$$\mathbf{C}_3 = \begin{array}{c|cc} & \mathrm{d}_s & \mathrm{q}_s \\ \hline \mathrm{d}_s & 1 & \\ \mathrm{m} & 1 & \\ \mathrm{n} & & 1 \\ \mathrm{q}_s & & 1 \end{array} \qquad \mathbf{C} = \mathbf{C}_1\cdot\mathbf{C}_2\cdot\mathbf{C}_3 = \begin{array}{c|cc} & \mathrm{d}_s & \mathrm{q}_s \\ \hline \mathrm{d}_s & a & \\ \mathrm{d}_r & \cos\alpha & -\sin\alpha \\ \mathrm{q}_r & \sin\alpha & \cos\alpha \\ \mathrm{q}_s & & a \end{array}$$

17.10

The resultant **C** is $\mathbf{C}_1\cdot\mathbf{C}_2\cdot\mathbf{C}_3$.

Rotation of the Stator Reference Frame

(*a*) It should be noted that, while a rotor layer of winding is assumed to be symmetrical around the circumference, on the *stator* the **d** winding (the **d** component of the layer) has different constants from the **q** winding. Hence, when the stator windings are shifted at an angle α, the rotor transformations have to be modified.

A stator winding **m** *shifted at an angle* α *(Fig. 17.12) has to be considered to lie on a separate layer from the other windings*, and it has to be derived from a primitive machine having an extra stator layer with a **d** and a **q** winding.

$$i^{ds2} = i'^{ds2} \qquad i^{ds1} = i'^{m} \cos \alpha \qquad i^{qs1} = i'^{m} \sin \alpha$$

$$\mathbf{C} = \begin{array}{c|c|c|} & \mathbf{d}_{s2} & \mathbf{m} \\ \hline \mathbf{d}_{s2} & 1 & \\ \hline \mathbf{d}_{s1} & & \cos\alpha \\ \hline \mathbf{q}_{s1} & & \sin\alpha \\ \hline \end{array} \qquad 17.11$$

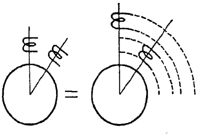


FIG. 17.12. Stator coil at an angle.

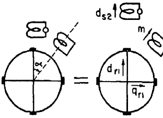


FIG. 17.13. Shaded-pole motor.

For instance, **C** of a shaded pole motor is (Fig. 17.13)

$$i^{ds2} = i'^{ds2} \qquad i^{ds1} = i'^{m} \cos \alpha \qquad i^{dr1} = i'^{dr1} \qquad i^{qr1} = i'^{qr1} \qquad i^{qs1} = i'^{m} \sin \alpha$$

$$\mathbf{C} = \begin{array}{c|c|c|c|c|} & \mathbf{d}_{s2} & \mathbf{m} & \mathbf{d}_{r1} & \mathbf{q}_{r1} \\ \hline \mathbf{d}_{s2} & 1 & & & \\ \hline \mathbf{d}_{s1} & & \cos\alpha & & \\ \hline \mathbf{d}_{r1} & & & 1 & \\ \hline \mathbf{q}_{r1} & & & & 1 \\ \hline \mathbf{q}_{s1} & & \sin\alpha & & \\ \hline \end{array} \qquad 17.12$$

(*b*) If the stator has a winding with the same constants along the **d** and **q** axes (as in a polyphase induction motor or alternator), then the reference frame on such a winding may be shifted in exactly the same way as on the rotor.

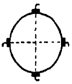
The Unit Transformation Tensor

(*a*) Many standard machines are identical with the primitive machine, containing various numbers of layers with various numbers of axes. For such machines the transformation tensor consists of the unit tensor having different numbers of axes, as shown in Table VI. Of course in such machines $\mathbf{Z}'$ is not found by $\mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C}$ but is simply picked out of $\mathbf{Z}$ of the primitive machine (Table V) by removing various rows and columns.

TABLE VI

ROTATING MACHINES WITH UNIT (OR DIAGONAL) TRANSFORMATION MATRIX

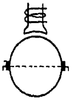
1



$$C = \begin{array}{c|cc} & d_r & q_r \\ \hline d_r & 1 & \\ q_r & & 1 \end{array}$$

Scherbius Advancer

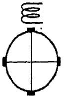
2



$$C = \begin{array}{c|cc} & d_s & q_r \\ \hline d_s & 1 & \\ q_s & & 1 \end{array}$$

D-C Shunt Motor

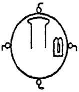
3



$$C = \begin{array}{c|ccc} & d_s & d_r & q_r \\ \hline d_s & 1 & & \\ d_r & & 1 & \\ q_r & & & 1 \end{array}$$

Single-Phase Induction Motor

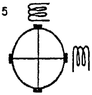
4



$$C = \begin{array}{c|ccc} & d_a & q_a & q_f \\ \hline d_a & 1 & & \\ q_a & & 1 & \\ q_f & & & 1 \end{array}$$

Reaction Motor

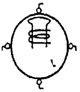
5



$$C = \begin{array}{c|cccc} & d_s & d_r & q_r & q_s \\ \hline d_s & 1 & & & \\ d_r & & 1 & & \\ q_r & & & 1 & \\ q_s & & & & a \end{array}$$

Split-Phase Induction Motor

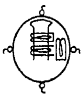
6



$$C = \begin{array}{c|ccc} & d_f & d_a & q_a \\ \hline d_f & 1 & & \\ d_a & & 1 & \\ q_a & & & 1 \end{array}$$

Alternator with No Amortisseur

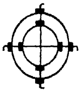
7



$$C = \begin{array}{c|ccccc} & d_{f2} & d_{f1} & d_a & q_a & q_f \\ \hline d_{f2} & 1 & & & & \\ d_{f1} & & 1 & & & \\ d_a & & & 1 & & \\ q_a & & & & 1 & \\ q_f & & & & & 1 \end{array}$$

Alternator with Amortisseur in Both Axes

8



$$C = \begin{array}{c|cccccc} & d_s & d_{r1} & d_{r2} & q_{r2} & q_{r1} & q_s \\ \hline d_s & 1 & & & & & \\ d_{r1} & & 1 & & & & \\ d_{r2} & & & 1 & & & \\ q_{r2} & & & & 1 & & \\ q_{r1} & & & & & 1 & \\ q_s & & & & & & 1 \end{array}$$

Double Squirrel Cage Induction Motor

For instance, for the single-phase induction motor (Fig. 17.14) $\mathbf{Z}'$ is found by simply removing the row and column of $\mathbf{q}_s$ from $\mathbf{Z}$ of the simpler primitive machine, equation 16.10.

$$\mathbf{Z}' = \begin{array}{c|ccc} & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r \\ \hline \mathbf{d}_s & r_s + L_s p & Mp & 0 \\ \mathbf{d}_r & Mp & r_r + L_r p & L_r p\theta \\ \mathbf{q}_r & -Mp\theta & -L_r p\theta & r_r + L_r p \end{array} \qquad 17.13$$

For the double squirrel-cage induction motor (under unbalanced operation, say a sudden short circuit on one of the stator phases), $\mathbf{Z}$ is

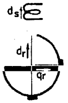


FIG. 17.14. Single-phase induction motor.

given in equations 16.30. (For balanced *polyphase* operation this $\mathbf{Z}$ is simplified as will be shown presently.)

(*b*) In some cases, such as the split-phase induction motor (Table VI-5), the unit tensor has to be multiplied by a turn-ratio tensor. That is, the diagonal units are replaced by constants.

A capacitor motor is the same as the split-phase (or asymmetrical) induction motor with a condenser $1/pC$ in series with axis $\mathbf{q}_s$ that, of course, is simply added to $L_{qs}p$ (without the intermediary step of $\mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C}$).

CHAPTER 18

PERFORMANCE CALCULATIONS

Calculation of the Currents

(*a*) In most machines the interconnection of coils has such a simple form that the $\mathbf{e}'$ *vector of the given network can be written down immediately* without the intermediary step of $\mathbf{e}' = \mathbf{C}_t \cdot \mathbf{e}$.

(*b*) The form of $\mathbf{Z}'$ depends on the components of the impressed voltage. The components of $\mathbf{e}'$ may in general assume three different forms:

1. In sudden short circuits they contain the Heaviside unit function **1**.

The $\mathbf{Z}'$ calculated by $\mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C}$ is used without any change. *In all machines with stationary reference axes* $\mathbf{Z}'$ is not a function of θ and $\mathbf{Y}' = \mathbf{Z}'^{-1}$ *can be solved with the aid of the expansion theorem without any further operational transformation* (such as shifting). That is, with the present method of attack the sudden short-circuit calculation of all rotating machines with stationary axes (if their speed $p\theta$ is maintained constant) is reduced to the simplicity of analysis of stationary networks with lumped resistances and inductances.

2. In a-c. steady state the components of $\mathbf{e}'$ contain complex numbers. In that case all p in $\mathbf{Z}'$ become $j\omega$, where ω is the frequency of the impressed voltage. Hence:

(*a*) All induced voltage terms become

$$pL = j\omega L_s = jX_s \quad \text{and} \quad pM = j\omega M = jX_m \qquad 18.1$$

(*b*) In all generated voltage terms, $p\theta$ becomes $v\omega$, where $v =$ (actual r.p.m.)/(syn. r.p.m.) and

$$p\theta L_{sd} = v\omega L_s = vX_s \quad \text{and} \quad p\theta M = v\omega M = vX_m \qquad 18.2$$

3. With d-c impressed voltages, the components of $\mathbf{e}'$ are constant and $p = 0$.

The currents in all cases are found by $\mathbf{i}' = \mathbf{Z}'^{-1} \cdot \mathbf{e}'$.

Calculation of Torque

(*a*) *The torque tensor* $\mathbf{G}'$ *may be established quickly by simply considering those components of* $\mathbf{Z}'$ *that contain* $p\theta$. In case of doubt $\mathbf{G}'$ is established from $\mathbf{G}$ of the primitive machine by $\mathbf{C}_t \cdot \mathbf{G} \cdot \mathbf{C}$.

(*b*) Once $\mathbf{i}'$ and $\mathbf{G}'$ have been calculated, then:

1. In sudden short-circuit or d-c. calculations the instantaneous torque is

$$f = \mathbf{i}' \cdot \mathbf{G}' \cdot \mathbf{i}' \tag{18.3}$$

2. In a-c. steady-state calculations the *steady* component of the torque is from $f = \mathbf{i}^* \cdot \mathbf{B}$ (in analogy with the definition of $P = \mathbf{i}^* \cdot \mathbf{e}$).

$$f = \text{real part of } \mathbf{i}^* \cdot \mathbf{G}\omega \cdot \mathbf{i} \tag{18.4}$$

The *oscillating* component is found if each component of $\mathbf{i}$ is substituted not as a complex number $i_1 + ji_2$ but as an instantaneous value $\sqrt{2}(i_1 \sin \omega t + i_2 \cos \omega t)$. The resulting expression will contain both steady and oscillating components.

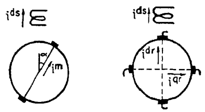
ω is introduced to express the torque in synchronous watts. The total torque is changed from synchronous watts T_{sw} to pound-feet T_{pf} by

$$T_{pf} = \frac{T_{sw} \times 33{,}000(\text{number of poles})}{2\pi(2 \times 60 \times \text{frequency})746} \tag{18.5}$$

Example of a Repulsion Motor

(*a*) As an example let the transient and steady-state equations of the repulsion motor (Fig. 18.1) be established. The transformation tensor is

$$\begin{aligned} i^{ds} &= i^{ds} \\ i^{dr} &= i^{a} \cos \alpha \\ i^{qr} &= i^{a} \sin \alpha \end{aligned} \qquad \mathbf{C} = \begin{array}{c|cc} & d_s & a \\ \hline d_s & 1 & \\ d_r & & \cos \alpha \\ q_r & & \sin \alpha \end{array} \tag{18.6}$$



(*a*) Repulsion motor. (*b*) Its primitive.

FIG. 18.1.

The $\mathbf{Z}$ of the primitive machine is (because of the smooth air gap $L_{dr} = L_{qr} = L_r$, etc.)

$$
\mathbf{Z} = \begin{array}{c|ccc}
 & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r \\ \hline
\mathbf{d}_s & r_s + L_s p & Mp & 0 \\
\mathbf{d}_r & Mp & r_r + L_r p & L_r p\theta \\
\mathbf{q}_r & -Mp\theta & -L_r p\theta & r_r + L_r p
\end{array}
\qquad
\mathbf{Z}\cdot\mathbf{C} = \begin{array}{c|cc}
 & \mathbf{d}_s & \mathbf{a} \\ \hline
\mathbf{d}_s & r_s + L_s p & Mp \cos\alpha\, p \\
\mathbf{d}_r & Mp & (r_r + L_r P)\cos\alpha + L_r \sin\alpha\, p\theta \\
\mathbf{q}_r & -Mp\theta & -L_r \cos\alpha\, p\theta + (r_r + L_r p)\sin\alpha
\end{array}
$$

M_p, r

$$
\mathbf{C}_t\cdot\mathbf{Z}\cdot\mathbf{C} = \mathbf{Z}' = \begin{array}{c|cc}
 & \mathbf{d}_s & \mathbf{a} \\ \hline
\mathbf{d}_s & r_s + L_s p & M\cos\alpha\, p \\
\mathbf{a} & M(\cos\alpha\, p - \sin\alpha\, p\theta) & r_r + L_r p
\end{array}
\qquad
\mathbf{G}' = \begin{array}{c|cc}
 & \mathbf{d}_s & \mathbf{a} \\ \hline
\mathbf{d}_s & & \\
\mathbf{a} & -M\sin\alpha &
\end{array}
\qquad 18.7
$$

The torque tensor is found by taking the coefficients of all $p\theta$.

(*b*) Let a unit function be impressed on the stator (that is, let the stator be suddenly short-circuited). Then

$$
\mathbf{e}' = \begin{array}{c|c}
\mathbf{d}_s & e1 \\
\mathbf{a} &
\end{array}
;\quad
\mathbf{Y}' = \mathbf{Z}'^{-1} = \begin{array}{c|cc}
 & \mathbf{d}_s & \mathbf{a} \\ \hline
\mathbf{d}_s & (r_r + L_r p)/D & -M\cos\alpha\, p/D \\
\mathbf{a} & -M(\cos\alpha\, p - \sin\alpha\, p\theta)/D & (r_s + L_s p)/D
\end{array}
\qquad 18.8
$$

where

$$D = (L_s L_r - M^2 \cos^2\alpha)p^2 + (r_r L_s + r_s L_r + M^2 \sin\alpha\cos\alpha\, p\theta)p + r_s r_r.$$

If the determinant is equated to zero, self-excited currents flow without the presence of an e.m.f. when the coefficient of the p term becomes zero. That may occur when α becomes sufficiently negative, so that

$$r_r L_s + r_s L_r = M^2 \sin\alpha\cos\alpha\, p\theta$$

With an applied e.m.f.

$$
\mathbf{i}' = \begin{array}{|cc|}
\mathbf{d}_s & \mathbf{a} \\ \hline
\dfrac{(r_r + L_r)p\, e1}{D} & \dfrac{r(\sin\alpha\, p\theta - \cos\alpha\, p)e1}{D}
\end{array}
=
\begin{array}{|cc|}
\mathbf{d}_s & \mathbf{a} \\ \hline
i^{ds} & i^{a}
\end{array}
\qquad 18.9
$$

Since α and $p\theta$ are constant, the currents can be solved by the expansion theorem.

Once the currents i^{ds} and i^a have been found, then the instantaneous torque is

$$f = \mathbf{i}'\cdot\mathbf{G}'\cdot\mathbf{i}' = -M\sin\alpha\, i^{ds}\, i^{a} \qquad 18.10$$

(*c*) When an a-c. terminal voltage is applied on the stator, then, replacing p by $j\omega$ and $p\theta$ by $v\omega$,

$$\mathbf{Z}' = \begin{array}{c|c|c|} & \mathbf{d}_s & \mathbf{a} \\ \mathbf{d}_s & r_s + jX_s & jX_m \cos\alpha \\ \mathbf{a} & X_m(j\cos\alpha - v\sin\alpha) & r_r + jX_r \end{array} \qquad \mathbf{e}' = \begin{array}{c|c|} \mathbf{d}_s & e \\ \mathbf{a} & \end{array}$$

$$\mathbf{Y}' = \mathbf{Z}'^{-1} = \begin{array}{c|c|c|} & \mathbf{d}_s & \mathbf{a} \\ \mathbf{d}_s & (r_r + jX_r)/D & -jX_m \cos\alpha/D \\ \mathbf{a} & X_m(\sin\alpha\, v - j\cos\alpha)/D & (r_s + jX_s)/D \end{array} \qquad 18.11$$

where

$$D = (r_r r_s + X_m^2 \cos^2\alpha - X_s X_r) + j(r_r X_s + r_s X_r + vX_m^2 \sin\alpha\cos\alpha).$$

$$\mathbf{i}' = \mathbf{Y}'\cdot\mathbf{e}' = \begin{array}{|c|c|} \mathbf{d}_s & \mathbf{a} \\ \hline \dfrac{(r_r + jX_r)e}{D} & \dfrac{X_m(\sin\alpha\, v - j\cos\alpha)e}{D} \end{array} = \begin{array}{|c|c|} \mathbf{d}_s & \mathbf{a} \\ \hline i^{ds} & i^a \end{array} \qquad 18.12$$

By $\mathbf{i}^*\cdot\omega\mathbf{G}\cdot\mathbf{i}$, the torque is the real part of

$$f = \frac{eX_m(\sin\alpha\, v + j\cos\alpha)}{D^*}\,(-X_m \sin\alpha)\,\frac{e(r_r + jX_r)}{D} \qquad 18.13$$

or the torque in synchronous watts is

$$f = \frac{e^2X_m^2 \sin\alpha\,(X_r\cos\alpha - r_r\sin\alpha\, v)}{(r_r r_s + X_m^2\cos^2\alpha - X_sX_r)^2 + (r_rX_s + r_sX_r + X_m^2\sin\alpha\cos\alpha\, v)^2} \qquad 18.14$$

It should be noted that no rationalization is necessary as D^*D is a real number.

Sign Convention of Central-Station Engineers

(*a*) The sign convention of synchronous-machine engineers differs from that of induction-motor engineers in the following respect:

1. The salient pole rotates instead of the armature; hence $p\theta$ has opposite sign.

2. Not the impressed voltage equation $\mathbf{e} = \mathbf{Z}\cdot\mathbf{i}$ is written but the generated voltage equation

$$\mathbf{e}_g = \mathbf{Z}_g\cdot\mathbf{i} \quad \text{or} \quad -\mathbf{e} = -\mathbf{Z}\cdot\mathbf{i} \qquad 18.15$$

$\mathbf{Z}_g$ of the primitive machine is given in Table V.

It is well to remember that:

(*a*) The right-hand side of the equation, $\mathbf{Z}_g \cdot \mathbf{i}$, represents all *internal* generated voltages of the machine in question.

(*b*) The left-hand side of the equation, $\mathbf{e}_g$, represents all voltages generated *external* to the machine in question. That is, $\mathbf{e}_g = \mathbf{Z}_g \cdot \mathbf{i}$ represents the relation:

External generated voltages = Internal generated voltages

3. The symbols e and E represent not impressed voltages but generated voltages, so that the components of $\mathbf{e}_g$ have positive signs (thereby those of $\mathbf{e}$ negative signs) as

$$\mathbf{e}_g = \begin{array}{cccc} \mathbf{d}_f & \mathbf{d}_a & \mathbf{q}_a & \mathbf{q}_f \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c|}{e_d} & \multicolumn{1}{c|}{e_q} & \multicolumn{1}{c|}{} \\ \hline \end{array} \qquad \mathbf{e} = -\mathbf{e}_g = \begin{array}{cccc} \mathbf{d}_f & \mathbf{d}_a & \mathbf{q}_a & \mathbf{q}_f \\ \hline \multicolumn{1}{|c|}{-E} & \multicolumn{1}{c|}{-e_d} & \multicolumn{1}{c|}{-e_q} & \multicolumn{1}{c|}{} \\ \hline \end{array} \qquad 18.16$$

Hence *the signs on both sides of their equations are the opposite* of those of the equations as they would have been written by induction-motor engineers.

4. The equations are written for the synchronous *generator* and not for the synchronous motor.

(*b*) In addition to the sign convention, the symbolisms of the engineers also differ. In particular, whereas induction-motor engineers use ohms and henries, synchronous-motor engineers use a per unit system.

In that system the unit of time is not the second but the time it takes for the field to describe 1 radian. This unit is $1/2\pi f$ part of the second; correspondingly all values of L in henries are multiplied by $2\pi f$. Because of the numerical identity of L and X, *in the per unit system inductances are denoted by X instead of L.*

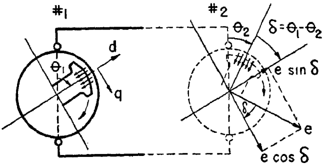
$\mathbf{e}_g$ Due to Infinite Bus

Since the armature axes $\mathbf{d}_a$ and $\mathbf{q}_a$ of a synchronous machine rotate, the armature components of $\mathbf{e}_g$, namely e_d and e_q (equation 18.16), *do not remain constant* as the load on the synchronous machine varies. The values of e_d and e_q depend on the system to which the machine is connected.

As one of the many possibilities, let an alternator (synchronous generator) be connected to an infinite bus. *An infinite bus may be considered an alternator whose armature impedance r_r, L_{dr}, L_{qr} is zero.*

It will be assumed that the field of the alternator *leads* the field of the bus by angle $\delta = \theta_1 - \theta_2$.

From Fig. 18.2 the total internal generated voltage in the armature of the bus is $e = -i^f Mp\theta$ and is due solely to its constant field excitation $-i^f$ (as its r and L are assumed to be zero). *Since* $\mathbf{e}_g$ *contains all*



(*a*) Alternator. (*b*) Infinite bus.

FIG. 18.2. Synchronous machine connected to infinite bus.

generated voltages that exist outside the given alternator, $e_d = e \sin \delta$, also $e_q = e \cos \delta$; hence

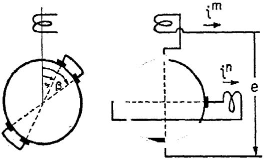
$$\begin{array}{cccc} d_f & d_a & q_a & q_f \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c|}{e \sin \delta} & \multicolumn{1}{c|}{e \cos \delta} & \multicolumn{1}{c|}{0} \\ \hline \end{array} \qquad 18.17$$

All three components are constant. At no load $\delta = 0$, and as the load on the alternator increases, δ increases. *When the generator becomes a motor,* δ *becomes negative.*

Since E along the field d_f is an external *generated* voltage, the current due to it, hence its field flux, is also negative, as shown in Fig. 18.2.

EXERCISES

1. Find C of the machines of Fig. 18.3.



(*a*) Repulsion motor. (*b*) Amplidyne.

FIG. :

2. Find the transient Z′ of the amplidyne. Find i^m and i^n. What is its torque in terms of i^m and i^n?

3. What are the Z and G tensors of the synchronous machine with no amortisseur winding?

4. Find the transient and steady-state Z′ and G′ of the repulsion motor in problem 1.

5. If the stator of the above repulsion motor is suddenly short-circuited (with $p\theta$ remaining constant), what are the transient currents i′ in the stator and rotor? What is the instantaneous torque?

CHAPTER 19

TRANSIENT STABILITY OF REGULATING DEVICES

Small Changes in Currents

(*a*) Interconnected rotating machines and stationary networks (in conjunction with mechanical devices) are used also in follow-up mechanisms and regulators where they are called upon to bring some disturbed system back into equilibrium. During this corrective period a small change of current Δi is superimposed upon a steady-state value. But, *as long as the speed of the rotating machines remains substantially constant, the equation of the corrective device can be written during the change as*

$$\Delta e = \mathbf{Z} \cdot \Delta i \qquad 19.1$$

where **Z** is calculated as shown hitherto. In such systems the determinant of the transient **Z** (containing p) may be investigated by Routh's criterion (to be shown presently) to find out whether the system is stable or unstable during the disturbance.

(*b*) When a regulator is used, the given system is divided into at least two parts: (1) the regulating device; (2) the system to be regulated. The **Z** of each of these may be established independently of the other's presence, then recombined into a resultant system.

Amplidyne Voltage Regulator

(*a*) Let **Z** of the voltage regulator of Fig. 19.1*a* be established, whose terminals *A–B* are connected to the armature of an alternator (through a rectifier) and terminals *C–D* are connected to the field of the same alternator.

The voltage-regulating device consists of an exciter whose field is influenced by an amplidyne controlled through the winding 2. (This is only an idealized representation of the actual control.) A transformer acts as a stabilizer.

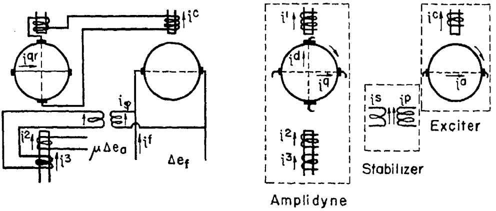
(*b*) The resultant regulator is divided into its component parts, the "primitive system," shown in Fig. 19.1*b*. The **Z** of the primitive system is

$$
\mathbf{Z} = \begin{array}{|c|c|c|}
\hline
\mathbf{Z}_1 & & \\
\hline
 & \mathbf{Z}_2 & \\
\hline
 & & \mathbf{Z}_3 \\
\hline
\end{array}
\tag{19.2}
$$

where

$$
\mathbf{Z}_1 = \begin{array}{c|c|c|c|c|c|}
 & 1 & 2 & 3 & d & q \\
\hline
1 & r_1 & & & & \\
\hline
2 & & r_2 & & & \\
\hline
3 & & & r_3 & & \\
\hline
d & & & & r_d & L_q p\theta_1 \\
\hline
q & -M_1 p\theta_1 & -M_2 p\theta_1 & -M_3 p\theta_1 & -L_d p\theta_1 & r_q \\
\hline
\end{array}
\tag{19.3}
$$

$$
\mathbf{Z}_2 = \begin{array}{c|c|c|}
 & p & s \\
\hline
p & r_p + L_p p & (M/n)p \\
\hline
s & (M/n)p & (r_s + L_s p)/n^2 \\
\hline
\end{array}
\qquad
\mathbf{Z}_3 = \begin{array}{c|c|c|}
 & c & a \\
\hline
c & r_c + L_c p & \\
\hline
a & -M_{ac} p\theta_3 & \\
\hline
\end{array}
$$



(a) Resultant system. (b) Primitive system.

FIG. 19.1. Amplidyne voltage regulator.

All induced voltages (p terms) of the amplidyne may be neglected in many applications, similarly r_a and L_a of the exciter armature.

The stabilizer constants r_p, L_p, and M are calculated on the primary side, the latter having n times the secondary turns. That is, with the use of the turn-ratio tensor $\mathbf{N}$, $\mathbf{Z}_2 = \mathbf{N}_t \cdot \mathbf{Z}'_2 \cdot \mathbf{N}$, where

$$
\mathbf{Z}_2' = \begin{array}{c|c|c|} & \mathrm{p}' & \mathrm{s}' \\ \hline \mathrm{p}' & r_p + L_p p & M_p \\ \hline \mathrm{s}' & M_p & r_s + L_s p \\ \hline \end{array}
\qquad
\mathbf{N} = \begin{array}{c|c|c|} & \mathrm{p} & \mathrm{s} \\ \hline \mathrm{p} & 1 & \\ \hline \mathrm{s} & & 1/n \\ \hline \end{array}
\qquad 19.4
$$

(*c*) The system of nine coils is interconnected into six meshes by

$$
\begin{array}{l}
i^1 = i^c \\
i^2 = i^2 \\
i^3 = i^3 \\
i^d = -i^c \\
i^q = i^q \\
i^p = i^p \\
i^s = i^3 \\
i^c = i^c \\
i^a = i^f + i^p
\end{array}
\qquad
\mathbf{C} = \begin{array}{c|c|c|c|c|c|c|} & \mathrm{f} & 2 & 3 & \mathrm{q} & \mathrm{c} & \mathrm{p} \\ \hline 1 & & & & & 1 & \\ \hline 2 & & 1 & & & & \\ \hline 3 & & & 1 & & & \\ \hline \mathrm{d} & & & & & -1 & \\ \hline \mathrm{q} & & & & 1 & & \\ \hline \mathrm{p} & & & & & & 1 \\ \hline \mathrm{s} & & & 1 & & & \\ \hline \mathrm{c} & & & & & 1 & \\ \hline \mathrm{a} & 1 & & & & & 1 \\ \hline \end{array}
= \begin{array}{|c|} \hline \mathbf{C}_1 \\ \hline \mathbf{C}_2 \\ \hline \mathbf{C}_3 \\ \hline \end{array}
\qquad 19.5
$$

(*d*) The resultant system is by

$$\mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C} = \mathbf{C}_{1t} \cdot \mathbf{Z}_1 \cdot \mathbf{C}_1 + \mathbf{C}_{2t} \cdot \mathbf{Z}_2 \cdot \mathbf{C}_2 + \mathbf{C}_{3t} \cdot \mathbf{Z}_3 \cdot \mathbf{C}_3 =$$

$$
\mathbf{Z}' = \begin{array}{c|c|c|c|c|c|c|} & \mathrm{f} & 2 & 3 & \mathrm{q} & \mathrm{c} & \mathrm{p} \\ \hline \mathrm{f} & & & & & -M_{oc} p\theta_3 & \\ \hline 2 & & r_2 & & & & \\ \hline 3 & & & r_3 + \frac{r_s}{n^2} + \frac{L_s}{n^2} p & & & (M/n)p \\ \hline \mathrm{q} & & -M_2 p\theta_1 & -M_3 p\theta_1 & r_q & (L_d - M_1)p\theta_1 & \\ \hline \mathrm{c} & & & & -L_q p\theta_1 & r_1 + r_d + r_c + L_c p & \\ \hline \mathrm{p} & & & (M/n)p & & -M_{ac} p\theta_3 & r_p + L_p p \\ \hline \end{array}
\qquad 19.6
$$

$$
\mathbf{e}' = \begin{array}{|c|c|c|c|c|c|} \mathrm{f} & 2 & 3 & \mathrm{q} & \mathrm{c} & \mathrm{p} \\ \hline \Delta e_f & \mu \Delta_e a & & & & \\ \hline \end{array}
$$

The stability of the system may be investigated by equating the determinant of Z to zero. Routh's (or other) criterion may be used for such studies.

Routh's Criterion

If the determinant of any transient $\mathbf{Z}$ is equated to zero, it can be arranged in descending powers of p as

$$a_5p^5 + a_4p^4 + a_3p^3 + a_2p^2 + a_1p^1 + a_0 = 0 \qquad 19.7$$

where the a's are *real* numbers.

The steps in determining the stability of the system are as follows:

1. Write down the coefficients *in pairs* as

$$\begin{array}{ccc} a_5 & a_3 & a_1 \\ a_4 & a_2 & a_0 \end{array} \qquad 19.8$$

2. Form the following products with the aid of the *first* column and each of the other columns.

$$b_1 = a_4a_3 - a_5a_2 \mid b_2 = a_4a_1 - a_5a_0$$

(as many such products as there are extra columns besides the first). Now three rows of coefficients are available.

$$\begin{array}{ccc} a_5 & a_3 & a_1 \\ a_4 & a_2 & a_0 \\ b_1 & b_2 & \end{array} \qquad 19.9$$

3. *Considering the last two rows only*, the previous product formation is repeated.

$$b_3 = b_1a_2 - a_4b_2 \mid b_4 = b_1a_0 \qquad 19.10$$

4. Considering again only the last two rows, the product formation is repeated until no more products can be formed from the last two rows.

Now, *if all the coefficients "a" or "b" are positive, the system is stable. If one of the coefficients is negative, the system is unstable.* An unstable condition indicates that, if an oscillation starts for any cause, it will not damp out but will increase in magnitude.

Usually one of the design constants is assumed to be variable and its limiting value is sought, which changes a stable system into an unstable one, or vice versa.

Time Constants and Amplification Factors

(a) In the transient-stability studies of control systems it is preferable to replace r and L by other types of constants, called "time constants," $T = L/r$, and "amplification factors," $\mu = Lp\theta/r$. For that purpose, in the equation of voltage

$$\mathbf{e} = \mathbf{R}\cdot\mathbf{i} + \mathbf{L}p\cdot\mathbf{i} + p\theta\,\mathbf{G}\cdot\mathbf{i} \qquad 19.11$$

let $\mathbf{i}$ be replaced by $\mathbf{R}\cdot\mathbf{i}$. That is, *let the resistance drops* $\mathbf{R}\cdot\mathbf{i}$ *be the variables, instead of the currents* $\mathbf{i}$. Since multiplication by the unit tensor

$I = R^{-1} \cdot R$ does not change the value of a tensor, equation 19.11 may be written as

$$\mathbf{e} = \mathbf{R}\cdot(\mathbf{R}^{-1}\cdot\mathbf{R})\cdot\mathbf{i} + \mathbf{L}p\cdot(\mathbf{R}^{-1}\cdot\mathbf{R})\cdot\mathbf{i} + p\theta\,\mathbf{G}\cdot(\mathbf{R}^{-1}\cdot\mathbf{R})\cdot\mathbf{i}$$

$$\mathbf{e} = (\mathbf{I} + \mathbf{L}\cdot\mathbf{R}^{-1}p + p\theta\,\mathbf{G}\cdot\mathbf{R}^{-1})\cdot\mathbf{R}\cdot\mathbf{i}$$

Let

$$\mathbf{L}\cdot\mathbf{R}^{-1} = \mathbf{T} = \text{time constant tensor} \qquad 19.12$$

$$p\theta\,\mathbf{G}\cdot\mathbf{R}^{-1} = \boldsymbol{\mu} = \text{amplification tensor} \qquad 19.13$$

Then the equation of voltage may be written in terms of them as

$$\mathbf{e} = (\mathbf{I} + \mathbf{T}p + \boldsymbol{\mu})\cdot\mathbf{R}\cdot\mathbf{i} \qquad 19.14$$

(*b*) Since $\mathbf{R}$ and $\mathbf{R}^{-1}$ are in general diagonal tensors, *multiplication with* $\mathbf{R}^{-1}$ *is equivalent to dividing each column of* $\mathbf{Z}$ *by the resistance in the diagonal term.* For instance, for equation 19.6,

$$\mathbf{Z}\cdot\mathbf{R}^{-1} = \begin{array}{c|cccccc} & \mathrm{f} & 2 & 3 & \mathrm{q} & \mathrm{c} & \mathrm{p} \\ \hline \mathrm{f} & & & & & -\frac{M_{ac}p\theta_3}{r_c'} & \\ 2 & & 1 & & & & \\ 3 & & & 1 + \frac{L_s p}{r_3' n^2} & & & \frac{M}{r_p n}p \\ \mathrm{q} & & -\frac{M_2 p\theta_1}{r_2} & -\frac{M_3 p\theta_1}{r_3'} & 1 & \frac{(L_d - M_1)p\theta_1}{r_c'} & \\ \mathrm{c} & & & & -\frac{L_q p\theta}{r_q} & 1 + \frac{L_c}{r_c'}p & \\ \mathrm{p} & & & \frac{M}{r_3' n}p & & -\frac{M_{ac}p\theta_3}{r_c'} & 1 + \frac{L_p}{r_p}p \end{array} \qquad 19.15$$

where $r_3' = r_3 + r_s/n^2$ and $r_c' = r_c + r_1 + r_d$. Introducing μ and T,

$$\mathbf{Z}'' = \begin{array}{c|cccccc} & \mathrm{f} & 2 & 3 & \mathrm{q} & \mathrm{c} & \mathrm{p} \\ \hline \mathrm{f} & & & & & -\mu_4 & \\ 2 & & 1 & & & & \\ 3 & & & 1 + T_1 p & & & T_4 p \\ \mathrm{q} & & -\mu_1 & -\mu_2 & 1 & \mu_5 & \\ \mathrm{c} & & & & -\mu_3 & 1 + T_3 p & \\ \mathrm{p} & & & T_2 p & & -\mu_4 & 1 + T_5 p \end{array} \qquad 19.16$$

$$e'' = \begin{array}{cccccc} \mathrm{f} & 2 & 3 & \mathrm{q} & \mathrm{c} & \mathrm{p} \\ \hline \Delta e_f & \mu\Delta e_a & & & & \end{array}$$

Since the row and column of q contains no T, it can be eliminated, thereby decreasing the number of μ's necessary to define the system.

Overall Amplification Factor

(*a*) If the last four rows and columns (on which no voltages are impressed) are eliminated by $Z' = Z_1 - Z_2 \cdot Z_4^{-1} \cdot Z_3$, the remaining terms can be written as

$$\Delta e_f = -\mu_0 \, \Delta e_a \tag{19.17}$$

where μ_0 is a function of Tp and μ. The equation shows how much a change Δe_a (impressed on the control field) is amplified by the time it passes through the regulator, and it also shows how much it is delayed during the passage. (An ideal regulator approaches infinite amplification and zero time delay.)

Eliminating the last four rows and rearranging, μ_0 may be expressed in the form

$$\Delta e_f = -\frac{\mu_a}{1 + T_a p + \dfrac{\mu_b \, T_b \, p}{(1 + T_c p)(1 + T_d p) - T_e T_f p^2}} \Delta e_a \tag{19.18}$$

(*b*) By various simplifying assumptions the degree of μ_0 in p may be decreased. For instance, if the leakage inductance of the stabilizer (and all inductances in series with them) is neglected, then in the denominator

$$(T_c T_d - T_c T_f)p^2 = 0 \tag{19.19}$$

and the degree of μ_0 in p decreases by 1.

(*c*) In the general case when the regulator is connected at several points to the system to be regulated, equation 19.17 is written as

$$\Delta \mathbf{e}_0 = \boldsymbol{\mu} \cdot \Delta \mathbf{e}_i \;\Big|\; \Delta e_\alpha = \mu_\alpha^m \, \Delta e_n \tag{19.20}$$

where $\boldsymbol{\mu} = \mu_\alpha^m$ is the *overall* amplification tensor representing the relation between the output and input voltages. (The previous $\boldsymbol{\mu}$ expressed the amplification of *each stage* of the regulator.)

In an amplifier the components of $\boldsymbol{\mu}$ are positive; in a regulator they are negative.

EXERCISES

1. Assume a shunt field along the short-circuited axis of the amplidyne voltage regulator, Fig. 19.2. Find **Z** of the whole system, **G** of the amplidyne, and the torque $\mathbf{i} \cdot \mathbf{G} \cdot \mathbf{i}$ in terms of the currents **i**.

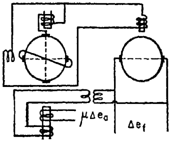


FIG. 19.2. Amplidyne voltage regulator.

2. Find the **T** and μ tensors.
3. What is the overall amplification factor of the system?

CHAPTER 20

ELIMINATION OF AXES

Calculation of Z' and e'

(*a*) Hitherto Z, e, and i had as many axes as the actual machine. In many machine problems (just as in stationary networks) attention is restricted to a few axes only. For instance, in a synchronous machine the phenomena, as viewed from the armature, are of primary importance; hence the field axes d_f and q_f may often be eliminated.

The elimination of axes is performed with exactly the same formulas as used before. If the axes of e_2 and i_2 (or e_1 and i_1) are eliminated, then

1. Z of the remaining axes is

$$Z_1' = Z_1 - Z_2 \cdot Z_4^{-1} \cdot Z_3 \mid Z_2' = Z_4 - Z_3 \cdot Z_1^{-1} \cdot Z_2 \qquad 20.1$$

2. e of the remaining axes is

$$e_1' = e_1 - Z_2 \cdot Z_4^{-1} \cdot e_2 \mid e_2' = e_2 - Z_3 \cdot Z_1^{-1} \cdot e_1 \qquad 20.2$$

so that the equation of voltage of the remaining axes is

$$e_1' = Z_1' \cdot i_1 \mid e_2' = Z_2' \cdot i_2 \qquad 20.3$$

3. When the current in the remaining axes has been found and later on the currents in the eliminated axes i_2 are wanted for some reason, they are found by

$$i_2 = Z_4^{-1} \cdot (e_2 - Z_3 \cdot i_1) \mid i_1 = Z_1^{-1} \cdot (e_1 - Z_2 \cdot i_2) \qquad 20.4$$

(*b*) In rotating machines it is often advantageous to place the term containing the eliminated voltages, namely, $-Z_3 \cdot Z_4^{-1} \cdot e_1 = -g_1 \cdot e_1$, not on the left-hand side but on the right-hand side of equation 20.3. Then the eliminated voltages are considered not part of a new impressed voltage e_2' but part of the new internal voltage $Z_4' \cdot i_2$, so that the new equations are written (in place of 20.3)

$$e_1' = Z_1' \cdot i_1 + g_2 \cdot e_2 \mid e_2' = Z_2' \cdot i_2 + g_1 \cdot e_1 \qquad 20.5$$

where

$$g_2 = Z_2 \cdot Z_4^{-1} \mid g_1 = Z_3 \cdot Z_1^{-1} \qquad 20.6$$

That is, the eliminated terminal voltages are assumed to influence the values of **R**, **L**, and G (or φ and B) of the machine but not the terminal voltage e of the remaining axes.

When some of the axes have been eliminated, the allowable transformations on the new system become restricted. In particular no new axes can be introduced that have a different velocity from the remaining axes. On the other hand, the remaining axes can be interconnected with other machines or can be shifted by a *constant* angle δ.

Calculation of G′ and B′

In rotating machines the question often arises, how to calculate the torque if some of the currents have been eliminated.

Only two special cases will be considered.

1. All stator (or field) currents $\mathbf{i}_1$ are eliminated. This special case is important in synchronous-machine studies.

2. All rotor (or armature) currents $\mathbf{i}_2$ are eliminated. This special case is important in induction-machine studies.

For this study the torque equation can be expressed as

$$f = \mathbf{i}_2^* \cdot \mathbf{B} = \mathbf{i}_2^* \cdot \mathbf{G} \cdot \mathbf{i} = \mathbf{i}_2^* \cdot \mathbf{G}_3 \cdot \mathbf{i}_1 + \mathbf{i}_2^* \cdot \mathbf{G}_4 \cdot \mathbf{i}_2 \qquad 20.7$$

1. When the *stator* (or field) current $\mathbf{i}_1$ is eliminated, its value is

$$\mathbf{i}_1 = \mathbf{Z}_1^{-1} \cdot (\mathbf{e}_1 - \mathbf{Z}_2 \cdot \mathbf{i}_2)$$

Substituting into the torque equation

$$f = \mathbf{i}_2^* \cdot (\mathbf{G}_4 - \mathbf{G}_3 \cdot \mathbf{Z}_1^{-1} \cdot \mathbf{Z}_2) \cdot \mathbf{i}_2 + \mathbf{i}_2^* \cdot \mathbf{G}_3 \cdot \mathbf{Z}_1^{-1} \cdot \mathbf{e}_1 \qquad 20.8$$

The expression $\mathbf{G}_4 - \mathbf{G}_3 \cdot \mathbf{Z}_1^{-1} \cdot \mathbf{Z}_2$ includes only those terms of the new $\mathbf{Z}'$ that contain $p\theta$, and $\mathbf{G}_3 \cdot \mathbf{Z}_1^{-1}$ includes only those terms of $\mathbf{g}$ that contain $p\theta$.

Hence *the new flux* $\mathbf{B}'$ *after elimination is again represented by the* $p\theta$ *terms of the new equation* $\mathbf{Z}_1' \cdot \mathbf{i}_1' - \mathbf{e}_1' = 0$ (*just as before elimination*). The torque is found now by $\mathbf{i}' \cdot \mathbf{B}'$ and not by $\mathbf{i}' \cdot \mathbf{g}' \cdot \mathbf{i}'$ since equation 20.8 cannot be so expressed.

2. When the rotor (or armature) current $\mathbf{i}_2$ is eliminated its value is

$$\mathbf{i}_2 = \mathbf{Z}_4^{-1} \cdot (\mathbf{e}_2 - \mathbf{Z}_3 \cdot \mathbf{i}_1) = \mathbf{Z}_4^{-1} \cdot \mathbf{e}_2 - \mathbf{A} \cdot \mathbf{i}_1$$

where

$$\mathbf{A} = \mathbf{Z}_4^{-1} \cdot \mathbf{Z}_3 \qquad 20.9$$

Substituting into the torque equation

$$f = (\mathbf{e}_2^* \cdot \mathbf{Z}_{4t}^{*-1} - \mathbf{i}_1^* \cdot \mathbf{A}_t^*) \mathbf{G}_3 \cdot \mathbf{i}_1 + \mathbf{G}_4 \cdot (\mathbf{Z}_4^{-1} \cdot \mathbf{e}_2 - \mathbf{A} \cdot \mathbf{i}_1) \qquad 20.10$$

This is the general formula for the calculation of torque.

Let the following special case that often occurs in induction motor studies be considered.

(*a*) The rotor has no impressed voltage, $\mathbf{e}_2 = 0$.
(*b*) The machine is smooth $\mathbf{i}_2^* \cdot \mathbf{G}_4 \cdot \mathbf{i}_2 = 0$.

Then

$$f = -\mathbf{i}_1^* \cdot \mathbf{A}_t^* \cdot \mathbf{G}_3 \cdot \mathbf{i}_1 \qquad 20.11$$

That is, *the new torque tensor is found from the old torque tensor by*

$$\mathbf{G}' = -\mathbf{A}_t^* \cdot \mathbf{G}_3 = -(\mathbf{Z}_4^{-1} \cdot \mathbf{Z}_3)_t^* \cdot \mathbf{G}_3 \qquad 20.12$$

Elimination of Field Axes of Alternators

(*a*) Let the $\mathbf{Z}_g$ and $\mathbf{e}_g$ tensors of an alternator with amortisseur windings $\mathbf{k}$ in both axes be given (equation 16.31). In order to eliminate the field (f) and amortisseur (k) axes $\mathbf{d}_f$, $\mathbf{d}_k$, and $\mathbf{q}_k$, let the order of the axes be changed to

$$\mathbf{Z}_g = \begin{array}{c|ccc|cc|} & \mathbf{d}_f & \mathbf{d}_k & \mathbf{q}_k & \mathbf{d}_a & \mathbf{q}_a \\ \hline \mathbf{d}_f & -r_f - L_f p & -M_{fk} p & & -M_{fd} p & \\ \mathbf{d}_k & -M_{fk} p & -r_{kd} - L_{kd} p & & -M_{kd} p & \\ \mathbf{q}_k & & & -r_{kq} - L_{kq} p & & -M_{kq} p \\ \hline \mathbf{d}_a & -M_{fd} p & -M_{kd} p & M_{kq} p\theta & -r - L_d p & L_q p\theta \\ \mathbf{q}_a & -M_{fd} p\theta & -M_{kd} p\theta & -M_{kq} p & -L_d p\theta & -r - L_q p \\ \hline \end{array} = \begin{array}{|c|c|} \hline \mathbf{Z}_1 & \mathbf{Z}_2 \\ \hline \mathbf{Z}_3 & \mathbf{Z}_4 \\ \hline \end{array} \qquad 20.13$$

$$\mathbf{e}_g = \begin{array}{c|c|} \mathbf{d}_f & E \\ \mathbf{d}_k & \\ \mathbf{q}_k & \\ \hline \mathbf{d}_a & e_d \\ \mathbf{q}_a & e_q \\ \end{array} = \begin{array}{|c|} \hline \mathbf{e}_1 \\ \hline \mathbf{e}_2 \\ \hline \end{array} \qquad 20.14$$

(Since the zero-sequence quantities remain unchanged throughout the following analysis, their equation is left out.)

After elimination the equations become

$$\mathbf{Z}_2' = \mathbf{Z}_4 - \mathbf{Z}_3 \cdot \mathbf{Z}_1^{-1} \cdot \mathbf{Z}_2 = \begin{array}{c|cc|} & \mathbf{d}_a & \mathbf{q}_a \\ \hline \mathbf{d}_a & -r - L_d(p)p & L_q(p)p\theta \\ \mathbf{q}_a & -L_d(p)p\theta & -r - L_q(p)p \\ \hline \end{array} = \mathbf{Z}_g' \qquad 20.15$$

$$\begin{aligned} \mathbf{e}_2' &= \mathbf{e}_2 - \mathbf{Z}_3 \cdot \mathbf{Z}_4^{-1} \cdot \mathbf{e}_1 = \\ &= \mathbf{e}_2 - \mathbf{g}_1 \cdot \mathbf{e}_1 \end{aligned} \quad \begin{array}{c|c|} \mathbf{d}_a & e_d - G(p)pE \\ \hline \mathbf{q}_a & e_q - G(p)p\theta E \end{array} = \mathbf{e}_g' \qquad 20.16$$

where *

$$L_d(p) = L_d - \frac{p^2(L_{kd}M_{fd}^2 - 2M_{kd}M_{fk}M_{fd} + L_fM_{kd}^2) + p(r_fM_{kd}^2 + r_{kd}M_{fd}^2)}{p^2(L_fL_{kd} - M_{fk}^2) + p(r_{kd}L_f + r_fL_{kd}) + r_{kd}r_f}$$

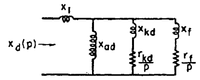
$$L_q(p) = L_q - \frac{M_{kq}^2 p}{r_{kq} + L_{kq}p} \qquad 20.17$$

$$G(p) = \frac{p(L_{kd}M_{fd} - M_{fk}M_{kd}) + r_{kd}M_{fd}}{p^2(L_fL_{kd} - M_{fk}^2) + p(r_{kd}L_f + r_fL_{kd}) + r_{kd}r_f}$$

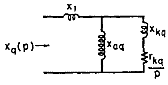
Hence *considering the armature axes only, their* **Z** *tensor* (also **R**, **L**, and **G**) *have exactly the same form after elimination of the field axes as before elimination, except that the open-circuit inductances L_d and L_q are now replaced by short-circuit (or "operational" or "transient") inductances* $L_d(p)$ and $L_q(p)$.

When each of the field axes has several windings on it, the above statement is still valid and $L_d(p)$ and $L_q(p)$ are the short-circuit impedances of the armature when looking toward the field. The direct and quadrature axes of the field then appear as stationary networks with several meshes.

(*b*) In design practice it is usually assumed that the three mutual inductances of the field, amortisseur, and armature are the same in the direct axis, that is $M_{fd} = M_{kd} = M_{fk}$ all denoted by x_{ad}. In that case $x_d(p)$ and $x_q(p)$ may be calculated from the equivalent circuits of Fig. 20.1 (where x_l is the armature, x_f the field, and x_{kd} the amortisseur *leakage* inductance).



(*a*) Direct axis.



(*b*) Quadrature axis.

FIG. 20.1. Calculation of $x_d(p) = L'_d$ and $x_q(p) = L'_q$

$G(p)$ is found by impressing E in series with x_f and calculating the difference of potential E' across x_{ad}. Then since $G(p)pE = E'$, therefore $G(p) = E'/pE$.

(*c*) Since by the sign convention of a synchronous machine

$$\mathbf{e} = \mathbf{R} \cdot \mathbf{i} + p\varphi + \mathbf{B}(-p\theta) \quad \text{or} \quad \mathbf{e}_g = -\mathbf{e} = -\mathbf{R} \cdot \mathbf{i} - p\varphi - \mathbf{B}(-p\theta) \qquad 20.18$$

* Crary and Waring, "The Operational Impedances of a Synchronous Machine," *General Electric Review*, Vol. 35, November, 1932, p. 578.

the new flux-density vector **B** is found as the coefficients of $p\theta$ in the equation $\mathbf{Z}_g \cdot \mathbf{i} - \mathbf{e}_g = 0$, while the new flux-linkage vector φ is found as the coefficients of all $-p$ terms.

$$\mathbf{B} = \begin{array}{c|c|} \mathrm{d}_a & L_q(p)i^q \\ \hline \mathrm{q}_a & -L_d(p)i^d + G(p)E \end{array} \qquad \varphi = \begin{array}{c|c|} \mathrm{d}_a & L_d(p)i^d - G(p)E \\ \hline \mathrm{q}_a & L_q(p)i^q \end{array} \tag{20.19}$$

The torque is by $\mathbf{i}\cdot\mathbf{B}$ (it cannot be expressed now as $\mathbf{i}\cdot\mathbf{G}\cdot\mathbf{i}$).

$$f = \mathbf{i}\cdot\mathbf{B} = i^d i^q[L_q(p) - L_d(p)] + i^q G(p)E \tag{20.20}$$

This is the torque exerted upon the armature (stationary member) by the currents and fluxes; hence it is the negative of the electromagnetic torque on the field. The expression is also equal to the *impressed mechanical torque driving the field* (rotating member), if the inertial force is ignored.

The Per Unit System of Central-Station Engineers*

(*a*) Central-station engineers denote the short-circuit inductances (since time is measured in radians) as

$$\begin{aligned} L_d(p) &= x_d(p) \qquad & r + L_d(p)p &= z_d(p) \\ L_q(p) &= x_q(p) \qquad & r + L_q(p)p &= z_q(p) \end{aligned} \tag{20.21}$$

and call them "transient" or "operational" impedances. Hence in per unit notation $\mathbf{Z}'_g$ and $\mathbf{e}'_g$ (equations 20.15 and 20.16) are written as

$$\mathbf{Z}'_g = \begin{array}{c|c|c|} & \mathrm{d} & \mathrm{q} \\ \hline \mathrm{d} & -z_d(p) & x_q(p)p\theta \\ \hline \mathrm{q} & -x_d(p)p\theta & -z_q(p) \end{array} \qquad \mathbf{e}'_g = \begin{array}{c|c|} \mathrm{d} & e_d - G(p)pE \\ \hline \mathrm{q} & e_q - G(p)p\theta E \end{array} \tag{20.22}$$

so that the equations $\mathbf{e}_g = \mathbf{Z}_g \cdot \mathbf{i}$ (or rather $\mathbf{e}_2 = \mathbf{g}_1 \cdot \mathbf{e}_1 + \mathbf{Z}_2 \cdot \mathbf{i}_2$) are written as

$$\begin{aligned} e_d &= G(p)pE - z_d(p)i_d + z_q(p)\,p\theta\,i_q \\ e_q &= G(p)\,p\theta\,E - x_d(p)\,p\theta\,i_d - z_q(p)i_q \\ e_0 &= -z_0 i_0 \end{aligned} \tag{20.23}$$

where the right-hand side of the equation contains all *internal* generated voltages and the left-hand side all *external* generated voltages.

* Park, "Two-Reaction Theory of Synchronous Machines," *Trans. A.I.E.E.*, April, 1929.

(b) The new flux linkage (the coefficients of all $-p$) and the new flux density (the coefficients of all $p\theta$) are

$$\boldsymbol{\varphi} = \begin{array}{c} \mathrm{d} \\ \mathrm{q} \end{array} \boxed{\begin{array}{c} x_d(p)i_d - G(p)E \\ \hline x_q(p)i_q \end{array}} = \boxed{\begin{array}{c} \varphi_d \\ \hline \varphi_q \end{array}}$$

$$\mathbf{B} = \begin{array}{c} \mathrm{d} \\ \mathrm{q} \end{array} \boxed{\begin{array}{c} x_q(p)i_q \\ \hline -x_d(p)i_d + G(p)E \end{array}} = \boxed{\begin{array}{c} B_d \\ \hline B_q \end{array}} = \boxed{\begin{array}{c} \varphi_q \\ \hline -\varphi_d \end{array}}$$

The torque driving the field is

$$T = f = \mathbf{i}\cdot\mathbf{B} = i_d i_q[x_q(p) - x_d(p)] + i_q G(p)E \qquad 20.24$$
$$= i_d B_d + i_q B_q = i_d \varphi_q - i_q \varphi_d$$

No Amortisseur Windings

Central-station engineers prefer to express $G(p)$, $L_d(p)$, and $L_q(p)$ in terms of the field time constant $T_0 = L_{df}/r_{df}$. For instance, in the absence of amortisseur windings

$$G(p) = \frac{M_d}{r_{fd} + L_{fd}p} = \frac{M_d/r_{fd}}{1 + (L_{fd}/r_{fd})p} = \frac{x_{ad}/r_{fd}}{1 + T_0 p} \qquad 20.25$$

$$x_d(p) = L_d(p) = L_d - \frac{M_d^2 p}{r_{fd} + L_{fd}p} = \frac{r_{fd}L_d + L_{fd}p(L_d - M_d^2/L_{fd})}{r_{fd} + L_{fd}p}$$

$$= \frac{L_d + (L_{fd}/r_{fd})p(L_d - M_d^2/L_{fd})}{1 + (L_{fd}/r_{fd})p} = \frac{L_d + T_0 p L_d'}{1 + T_0 p}$$

$$= \frac{x_d' T_0 p + 1}{T_0 p + 1} \qquad 20.26$$

where

$$x_d' = L_d - M_d^2/L_{fd} = \text{short-circuit inductance} \qquad 20.27$$

if $r_{fd} = 0$ (or $p = \infty$).

In the absence of amortisseurs it is also convenient to call E not the actual field terminal voltage E_{fd} but the armature generated voltage $i^f x_{ad}$. That is,

$$E = \frac{E_{fd}}{r_{fd}} x_{ad} \quad \text{so that} \quad G(p) = \frac{1}{T_0 p + 1}$$

With no amortisseur $x_q(p) = x_q$.

Park's Sign Convention of Flux Linkages

(*a*) While in these pages the definition of φ and $\mathbf{B}$ has been introduced in order to write for the primitive machine $\mathbf{e} = \mathbf{R}\cdot\mathbf{i} + p\varphi + \mathbf{B}p\theta$, hence to write for the synchronous machine $\mathbf{e}_g = -\mathbf{e} = -\mathbf{R}\cdot\mathbf{i} - p\varphi - \mathbf{B}(-p\theta)$, Park on the other hand writes for the synchronous machine

$$\mathbf{e}_g = -\mathbf{R}\cdot\mathbf{i} + p\mathbf{\Psi} + \boldsymbol{\gamma}\cdot\mathbf{\Psi}p\theta \qquad \boldsymbol{\gamma} = \begin{array}{c|cc} & \mathrm{d} & \mathrm{q} \\ \hline \mathrm{d} & & -1 \\ \mathrm{q} & 1 & \end{array} \qquad 20.28$$

That is, *Park's flux-linkage vector* $\mathbf{\Psi}$ *is the negative of* φ *as defined here, and he does not introduce the concept of flux-density vector* $\mathbf{B}$.

$$\mathbf{\Psi} = \begin{array}{c|c} \mathrm{d} & G(p)E - x_d(p)i_d \\ \mathrm{q} & -x_q(p)i_q \end{array} = \begin{array}{|c|} \psi_d \\ \psi_q \end{array} \qquad \boldsymbol{\gamma}\cdot\mathbf{\Psi} = \begin{array}{|c|} -\psi_q \\ \psi_d \end{array} \qquad 20.29$$

The relation between $\mathbf{\Psi}$ and $\mathbf{B}$ is

$$\mathbf{\Psi} = \begin{array}{c|c} \mathrm{d} & \psi_d \\ \mathrm{q} & \psi_q \end{array} = \begin{array}{|c|} B_q \\ -B_d \end{array} \quad \text{and} \quad \mathbf{B} = \begin{array}{c|c} \mathrm{d} & B_d \\ \mathrm{q} & B_q \end{array} = \begin{array}{|c|} -\psi_q \\ \psi_d \end{array} \quad \text{so that } \mathbf{B} = \boldsymbol{\gamma}\cdot\mathbf{\Psi} \qquad 20.30$$

(*b*) In terms of ψ_d and ψ_q, equations 20.23 are written as

$$\begin{aligned} e_d &= -ri_d + p\psi_d - \psi_q p\theta \\ e_q &= -ri_q + p\psi_q + \psi_d p\theta \\ e_0 &= -ri_0 + p\psi_0 \end{aligned} \qquad 20.31$$

The torque equation is written by Park as $\mathbf{\Psi} \times \mathbf{i}$ (the cross-product of conventional vector analysis)

$$T = f = i_q\psi_d - i_d\psi_q \qquad 20.32$$

representing the torque on the armature.

Steady-State Performance of Synchronous Machines

(*a*) At synchronous speed $p = 0$ and $p\theta =$ unity. Then $x_d(p) = x_d$, $x_q(p) = x_q$, $G(p) = 1$, if E is the internal generated voltage. On an infinite bus

$$\mathbf{Z}_g = \begin{array}{c|cc} & d & q \\ \hline d & -r & x_q \\ q & -x_d & -r \end{array} \qquad \mathbf{e}_g = \begin{array}{c|c} d & e \sin \delta \\ q & e \cos \delta - E \end{array} \qquad \mathbf{B} = \begin{array}{c|c} d & x_q i_q \\ q & -x_d i_d + E \end{array} \qquad 20.33$$

$$\mathbf{Y} = \mathbf{Z}_g^{-1} = \begin{array}{c|cc} & d & q \\ \hline d & -r/D & -x_q/D \\ q & x_d/D & -r/D \end{array} \qquad \mathbf{i} = \begin{array}{c|c} d & [-re \sin \delta - x_q(e \cos \delta - E)]/D \\ q & [e \sin \delta\, x_d - r(e \cos \delta - E)]/D \end{array} \qquad 20.34$$

where $D = r^2 + x_d x_q$. The mechanical torque driving the field (or the electromagnetic torque on the armature) is

$$T = f = \mathbf{i} \cdot \mathbf{B} = i_d i_q (x_q - x_d) + E i_q \qquad 20.35$$

(*b*) It should be remembered that i^d and i^q are *hypothetical* currents (constant in value during steady state) and may be assumed to exist inside the armature as measured by an observer who rotates with the field poles. The *actual* armature currents flowing out of the stationary terminals are sinusoidal currents i^a and i^b that may be found from i^d and i^q by a simple transformation to be shown in equation 27.14.

The reason for finding i^d and i^q first is that the equations for them are simple, while those that contain i^a and i^b are more involved.

Synchronous Machine Running below Synchronism

When a synchronous machine is connected to an infinite bus, Fig. 18.2, but runs below synchronism at a speed of $p\theta = v\omega$ (or at a slip of $s = 1 - v$), then $\delta = s\omega t$. When the field excitation is removed, all currents are of slip frequency, in $\mathbf{Z}$ of equation 20.22 all p become $js\omega$ and all $p\theta$ become $v\omega$. If, in equation 18.17, $e \sin s\omega t = \hat{e} = e/\sqrt{2}$ and $e \cos s\omega t = -j\hat{e}$, then

$$\mathbf{Z}_g = \begin{array}{c|cc} & d & q \\ \hline d & -r - jsx_d(js) & vx_q(js) \\ q & -vx_d(js) & -r - jsx_q(js) \end{array} \qquad \mathbf{e}_g = \begin{array}{|c|} \hat{e} \\ -j\hat{e} \end{array} \qquad \omega\mathbf{G} = \begin{array}{c|cc} & d & q \\ \hline d & & x_q(js) \\ q & -x_d(js) & \end{array} \qquad 20.36$$

$x_d(js)$ and $x_q(js)$ are calculated from the equivalent circuit of Fig. 20.1, where p is replaced by js, each resistance becomes $-jr/s$ so that $x_d(js)$ has the form $a - jb$. For every slip a different resistance value exists.

The currents are found by $\mathbf{i} = \mathbf{Z}_g^{-1} \cdot \mathbf{e}_g$, and the constant torque by the real part of $\mathbf{i}^* \cdot \omega\mathbf{G} \cdot \mathbf{i}$.

$$f_c = \text{Real of } i^{d*} x_q(js) i^q - i^{q*} x_d(js) i^d \qquad 20.37$$

The oscillating torques are found by substituting for **i** the instantaneous values $i = \sqrt{2}\,(i_1 \sin s\omega t + i_2 \cos s\omega t)$ instead of complex numbers.

When the amortisseur winding is absent, the equations represent (assuming smooth air gap) a two-phase induction motor running with a single-phase rotor (Fig. 20.2).

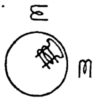


FIG. 20.2. Two-phase induction motor with single-phase rotor.

The Interconnection of Synchronous Machines *

(*a*) The concept of "interconnection of axes" implies the interconnection of *physical* axes, such as brushes, slip rings, stator windings, etc. When the axes are *hypothetical*, such as the $\mathbf{d}_a$ and $\mathbf{q}_a$ axes of synchronous machine armatures, their interconnection involves two steps:

1. The actually existing axes **a** and **b** of the armature are interconnected by a **C**.
2. The **a** and **b** axes of **C** are transformed into the hypothetical axes **d** and **q** by equation 6.11 so that a new **C′** represents the interconnection of the hypothetical axes.

(*b*) When two interconnected synchronous machines (Fig. 20.3) run at the same speed with the rotor of the second machine lagging behind

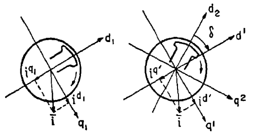


FIG. 20.3. The interconnection of hypothetical axes.

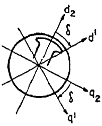


FIG. 20.4. Rotating the reference frame by a constant angle δ.

that of the first by an angle δ (the value of δ depending on the load) the *hypothetical* axes **d** and **q** may be interconnected in one step by noting that *the resultant current vectors* **i** *in the armatures of both machines are equal and have the same direction in space at each instant.* Hence *if in the second machine new reference axes* **d′** *and* **q′** *are introduced parallel to those of the first machine* $\mathbf{d}_1$ *and* $\mathbf{q}_1$, then the components of **i** are equal along the reference axes and *the latter can be connected in series.*

* Doherty and Nickle, "Synchronous Machines, II," *Trans. A.I.E.E.*, 1926.

The transformation tensor rotating the reference axes $\mathbf{d}_2$ and $\mathbf{q}_2$ of Fig. 20.4 to $\mathbf{d}'$ and $\mathbf{q}'$ by an angle δ (the same as in brush rotation, equation 17.4) is

$$\mathbf{C}_1 = \begin{array}{c|cc} & \mathbf{d}' & \mathbf{q}' \\ \hline \mathbf{d}_2 & \cos\delta & -\sin\delta \\ \mathbf{q}_2 & \sin\delta & \cos\delta \end{array} \qquad 20.38$$

By $\mathbf{C}_t \cdot \mathbf{Z}_g \cdot \mathbf{C}$ and $\mathbf{C}_t \cdot \mathbf{e}_g$, equations 20.22 become

$$\mathbf{Z}'_g = \begin{array}{c|cc} & \mathbf{d}' & \mathbf{q}' \\ \hline \mathbf{d}' & \begin{array}{l} -r - [x_d(p)\cos^2\delta + x_q(p)\sin^2\delta]p + \\ + [x_q(p) - x_d(p)]p\theta \sin\delta\cos\delta \end{array} & \begin{array}{l} [x_q(p)\cos^2\delta + x_d(p)\sin^2\delta]\,p\theta + \\ + [x_d(p) - x_q(p)]\sin\delta\cos\delta p \end{array} \\ \mathbf{q}' & \begin{array}{l} -[x_d(p)\cos^2\delta + x_q(p)\sin^2\delta]p\theta + \\ + [x_d(p) - x_q(p)]\sin\delta\cos\delta p \end{array} & \begin{array}{l} -r - [x_q(p)\cos^2\delta + x_d(p)\sin^2\delta]p + \\ + [x_d(p) - x_q(p)]p\theta \sin\delta\cos\delta \end{array} \end{array}$$

20.39

$$\mathbf{e}'_g = \begin{array}{c|c} \mathbf{d}' & e'_d - [\cos\delta\, p - \sin\delta\, p\theta]G(p)E \\ \mathbf{q}' & e'_q + [\sin\delta\, p - \cos\delta\, p\theta]G(p)E \end{array}$$

The coefficients of the $p\theta$ terms of $(\mathbf{Z}' \cdot \mathbf{i}' - \mathbf{e}'_g)$ give $\mathbf{B}'$.

(*c*) The transformation tensor that interconnects the hypothetical axes of two synchronous machines is

$$\begin{aligned} i^{d1} &= i^{d1} \\ i^{q1} &= i^{q1} \\ i^{d'} &= -\,i^{d'} \\ i^{q'} &= -\,i^{q'} \end{aligned} \qquad \mathbf{C}_2 = \begin{array}{c|cc} & \mathbf{d}_1 & \mathbf{q}_1 \\ \hline \mathbf{d}_1 & 1 & \\ \mathbf{q}_1 & & 1 \\ \mathbf{d}' & -1 & \\ \mathbf{q}' & & -1 \end{array} \qquad 20.40$$

(*d*) The shifting of axes and the interconnection of two machines may be performed in one step as

$$\mathbf{C} = \mathbf{C}_1 \cdot \mathbf{C}_2 = \begin{array}{c|cc} & \mathbf{d}_1 & \mathbf{q}_1 \\ \hline \mathbf{d}_1 & 1 & \\ \mathbf{q}_1 & & 1 \\ \mathbf{d}_2 & -\cos\delta & \sin\delta \\ \mathbf{q}_2 & -\sin\delta & -\cos\delta \end{array} \qquad 20.41$$

EXERCISES

1. Derive the value of $x_d(p)$, $G(p)$, $x_q(p)$ (equations 20.25–20.27) when no amortisseur winding exists on the synchronous machine. (Start with the original three equations and eliminate the field axis.)

2. Given the steady-state **Z** and **e** of two salient-pole synchronous machines, that is, $\mathbf{Z}_1$, $\mathbf{Z}_2$ and $\mathbf{e}_1$, $\mathbf{e}_2$. What are the resultant $\mathbf{Z}'$ and $\mathbf{e}'$ of the interconnected system when

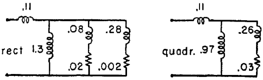


FIG. 20.5.

the second machine lags behind the first machine by angle δ? What is the torque of each machine?

3. The direct and quadrature-axis quantities of a salient-pole synchronous machine are given (in per unit) in Fig. 20.5.

(*a*) What are $x_d(p)$ and $x_q(p)$?

(*b*) What are $x_d(js)$ and $x_q(js)$ for $s = 1, 0.75, 0.5, 0.25, 0$?

(*c*) If $r = 0.015$ and $e = 1$, find i^d, i^q and the torque at the above slips.

(*d*) Find x_d and x_q.

(*e*) If $E = 1.1$ and $e = 1$, find the steady-state currents i^d and i^q and the torque for $\delta = 0°, 30°, 60°, 90°, 120°, 150°, 180°$.

(*f*) When running at synchronous speed on open circuit, the armature is suddenly short-circuited ($e_q = -1$). What are the instantaneous currents and torques?

CHAPTER 21

THE REVOLVING-FIELD THEORY

Transformations Necessary to Establish Equivalent Circuits

(*a*) The study of rotating machinery and the understanding of their physical behavior are facilitated by two artifices:

1. Locus diagrams.
2. Equivalent circuits.

A systematic study of locus diagrams by tensorial concepts has been undertaken elsewhere.* The tensorial method of attack offers also a powerful aid in establishing a group of stationary networks whose performance parallels practically any type of standard rotating machine, as far as steady-state behavior and small oscillations are concerned. Besides facilitating the visualization of physical phenomena taking place inside a rotating machine and offering computational help, an equivalent circuit also permits the determination of the steady-state and hunting performance with the aid of the a-c. network analyzer.

(*b*) For any particular machine the equivalent circuit is established by finding a transformation matrix **C** that changes the asymmetrical **Z** into a symmetrical one. Three such transformations may be mentioned here:

1. The method of symmetrical components.
2. The rotation of the reference axes by a constant or variable angle δ.
3. Division of an equation of voltage by a quantity.

Representation of Torque on the Equivalent Circuits

A rather large number of equivalent circuits possess the disadvantage of not indicating the torque. Even those that do show the torque require an elaborate derivation to prove the correctness of the representation. Makeshift schemes such as subtracting the losses from the input have no more value as aids for visualization or computation than

* *A.T.E.M.*, p. 160.

the equations themselves. The tensorial method of attack makes a clean sweep of this difficulty.

As the torque is $\mathbf{i}\cdot\mathbf{B}$ (where $\mathbf{B}$ is the resultant rotor flux density $\mathbf{G}\cdot\mathbf{i}$ and where $\mathbf{G}$ contains inductances) *by virtue of* $\mathbf{G}$, *also* $\mathbf{i}$ *and hence* $\mathbf{G}\cdot\mathbf{i} = \mathbf{B}$ *all being tensors,* $\mathbf{B}$ *must appear on any logical equivalent circuit as a measurable quantity,* in particular as sets of differences of potential $\mathbf{E}$. Similarly, the torque

$$f = \mathbf{i}^*\cdot\mathbf{B} = \mathbf{i}^*\cdot\mathbf{E} = i^{1*}E_1 + i^{2*}E_2 + \cdots \qquad 21.1$$

must be a quantity to be measured by adding up the indicated wattmeter readings. The components of $\mathbf{E} = \mathbf{B}$ are to be determined by tracing out the voltage drops $\mathbf{G}\cdot\mathbf{i}$ on the equivalent network.

Forward- and Backward-Revolving Fields

(*a*) When on a layer of winding there are two axes at right angles in space (say $\mathbf{d}$ and $\mathbf{q}$) each containing a-c. currents i^d and i^q of the same frequency, then each alternating current may be divided into a hypothetical forward- and a backward-rotating component by the method of two-phase symmetrical components (equation 9.11)

$$\begin{aligned} i^d &= (i^1 + i^2)/2 \\ i^q &= -j(i^1 - i^2)/2 \end{aligned} \qquad \mathbf{C} = \tfrac{1}{2}\;\begin{array}{c|cc} & 1 & 2 \\ \hline \mathrm{d} & 1 & 1 \\ \mathrm{q} & -j & j \end{array} \qquad \mathbf{C}_t^* = \tfrac{1}{2}\;\begin{array}{c|cc} & \mathrm{d} & \mathrm{q} \\ \hline 1 & 1 & j \\ 2 & 1 & -j \end{array} \qquad 21.2$$

The axes 1 and 2 represent the reference frame of the revolving-field theory; the axes d and q represent the reference frame of the cross-field theory. *With the aid of the above* $\mathbf{C}$ *and its inverse* (one such $\mathbf{C}$ for each layer of winding), *the equations of one theory can be converted into those of the other by routine manipulations.*

In converting the equations of the two theories into each other with the aid of $\mathbf{C}$ it is important to remember that there should exist as many equations as there are physical axis. If some of the axes have already been eliminated by $\mathbf{Z}_1 - \mathbf{Z}_2\cdot\mathbf{Z}_4^{-1}\cdot\mathbf{Z}_3$, the two sets of equations cannot be transformed into each other by the given $\mathbf{C}$.

(*b*) As in stationary networks, *the above* $\mathbf{C}$ *is valid only for the primitive machine.* If the axes have different numbers of turns or are at an angle α or are interconnected with other coils, the above $\mathbf{C}$ has to be modified either by the steps shown previously or by a method equivalent to those steps.

The use of the above **C** eliminates certain components of **Z** (reduces **Z** to a diagonal form) only if r and L along the **d** and **q** axes are the same, in particular only if **Z** has the form

$$\mathbf{Z} = \begin{array}{c|cc} & \mathbf{d} & \mathbf{q} \\ \hline \mathbf{d} & Z & Z_1 \\ \mathbf{q} & -Z_1 & Z \end{array} \qquad \mathbf{C}_t^*\cdot\mathbf{Z}\cdot\mathbf{C} = \tfrac{1}{2}\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & Z - jZ_1 & \\ 2 & & Z + jZ_1 \end{array} \tag{21.3}$$

Such a case occurs in the rotor windings of machines with smooth air gap where

$$\mathbf{Z} = \begin{array}{c|cc} & \mathbf{d}_r & \mathbf{q}_r \\ \hline \mathbf{d}_r & r_r + jX_r & X_r v \\ \mathbf{q}_r & -X_r v & r_r + jX_r \end{array} \qquad \mathbf{Z}' = \tfrac{1}{2}\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & r_r + (1 - v)jX_r & 0 \\ 2 & 0 & r_r + (1 + v)jX_r \end{array} \tag{21.4}$$

The real advantage of the use of this **C** shows up in the calculation of torque.

Two-Phase Induction Motor with Unbalanced Voltages

(*a*) Let *unbalanced* voltages be impressed on the stator of a balanced two-phase induction motor. Since its **C** is the unit tensor (Table V*a*), replacing p by $j\omega$ and $p\theta$ by $v\omega$,

$$\mathbf{Z} = \begin{array}{c|cccc} & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\ \hline \mathbf{d}_s & r_s + jX_s & jX_m & & \\ \mathbf{d}_r & jX_m & r_r + jX_r & X_r v & X_m v \\ \mathbf{q}_r & -X_m v & -X_r v & r_r + jX_r & jX_m \\ \mathbf{q}_s & & & jX_m & r_s + jX_s \end{array}$$

$$\omega\mathbf{G} = \begin{array}{c|cccc} & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\ \hline \mathbf{d}_s & & & & \\ \mathbf{d}_r & & & X_r & X_m \\ \mathbf{q}_r & -X_m & -X_r & & \\ \mathbf{q}_s & & & & \end{array} \qquad \mathbf{e} = \begin{array}{cccc} \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\ \hline e_{ds} & & & e_{qs} \end{array} \tag{21.5}$$

The equations of the cross-field theory are $\mathbf{e} = \mathbf{Z}\cdot\mathbf{i}$ and f = Real of $\mathbf{i}^*\cdot\omega\mathbf{G}\cdot\mathbf{i}$.

(*b*) Let revolving axes 1 and 2 be introduced on both stator and rotor (Fig. 21.1).

$$
\begin{aligned}
i^{ds} &= (i^{1s} + i^{2s})/2 \\
i^{dr} &= (i^{1r} + i^{2r})/2 \\
i^{qr} &= -j(i^{1r} - i^{2r})/2 \\
i^{qs} &= -j(i^{1s} - i^{2s})/2
\end{aligned}
\qquad
\mathbf{C} = \tfrac{1}{2}\;
\begin{array}{c|cccc|}
 & \mathbf{1}_s & \mathbf{1}_r & \mathbf{2}_s & \mathbf{2}_r \\
\hline
\mathbf{d}_s & 1 & & 1 & \\
\mathbf{d}_r & & 1 & & 1 \\
\mathbf{q}_r & & -j & & j \\
\mathbf{q}_s & -j & & j & \\
\hline
\end{array}
\tag{21.6}
$$

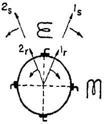


FIG. 21.1. Forward- and backward-revolving axes.

By $\mathbf{C}_t^* \cdot \mathbf{Z} \cdot \mathbf{C}$ and $\mathbf{C}_t^* \cdot \mathbf{E}$

$$
\mathbf{Z}' = \tfrac{1}{2}\;
\begin{array}{c|cc|cc|}
 & \mathbf{1}_s & \mathbf{1}_r & \mathbf{2}_s & \mathbf{2}_r \\
\hline
\mathbf{1}_s & r_s + jX_s & jX_m & & \\
\mathbf{1}_r & jX_m(1 - v) & r_r + jX_r(1 - v) & & \\
\hline
\mathbf{2}_s & & & r_s + jX_s & jX_m \\
\mathbf{2}_r & & & jX_m(1 + v) & r_r + jX_r(1 + v) \\
\hline
\end{array}
\tag{21.7}
$$

$$
\mathbf{G}' = \tfrac{1}{2}\;
\begin{array}{c|cc|cc|}
 & \mathbf{1}_s & \mathbf{1}_r & \mathbf{2}_s & \mathbf{2}_r \\
\hline
\mathbf{1}_s & & & & \\
\mathbf{1}_r & -jX_m & -jX_r & & \\
\hline
\mathbf{2}_s & & & & \\
\mathbf{2}_r & & & jX_m & jX_r \\
\hline
\end{array}
\tag{21.8}
$$

$$
\mathbf{e}' = \tfrac{1}{2}\;
\begin{array}{|cc|cc|}
\mathbf{1}_s & \mathbf{1}_r & \mathbf{2}_s & \mathbf{2}_r \\
\hline
e_{ds} + je_{qs} & & e_{ds} - je_{qs} & \\
\hline
\end{array}
\tag{21.9}
$$

Each revolving field acts as if the other were not present. Note in $\mathbf{Z}'$ that no mutuals exist between axes 1 and 2.

(c) If in **Z** and **e** the row 1_r is divided by $1 - v$ and the row 2_r by $1 + v$ (the currents remain thereby unchanged),

$$\mathbf{Z}' = \begin{array}{c|c|c|c|c|} & 1_s & 1_r & 2_s & 2_r \\ \hline 1_s & r_s + jX_s & jX_m & & \\ \hline 1_r & jX_m & \frac{r_r}{1 - v} + jX_r & & \\ \hline 2_s & & & r_s + jX_s & jX_m \\ \hline 2_r & & & jX_m & \frac{r_r}{1 + v} + jX_r \\ \hline \end{array} \qquad 21.10$$

$$\mathbf{E}' = \omega\mathbf{G}'\cdot\mathbf{i}' = \begin{array}{c|c|} 1_s & \\ \hline 1_r & -(jX_m i^{1s} + jX_r i^{1r}) \\ \hline 2_s & \\ \hline 2_r & jX_m i^{2s} + jX_r i^{2r} \\ \hline \end{array} = \begin{array}{c|c|} 1_s & \\ \hline 1_r & E_{1r} \\ \hline 2_s & \\ \hline 2_r & E_{2r} \\ \hline \end{array} \qquad 21.11$$

$$f = \text{Real of } \mathbf{i}'^{*}\cdot\mathbf{E}' = R(i^{1r*}E_{1r} + i^{2r*}E_{2r}) = W_{1r} + W_{2r} \qquad 21.12$$

As **Z** is symmetrical, its equivalent circuit may be established as shown in Fig. 21.2 ($X_s = x_s + X_m$ and $X_r = x_r + X_m$). The two sequence networks are independent. The torques are measured by two

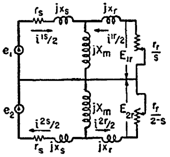


FIG. 21.2. Equivalent circuit of an induction motor on unbalanced voltages.

wattmeter readings, representing the difference in the rotor losses of the two sequence networks.

It is customary to leave out the ½ in **Z**′ and **G**′ (but not in **e**′). In that case the currents are half of the shown value and f is the torque per phase.

(*d*) Since no e.m.f. is impressed on the rotor, the rotor axes $\mathbf{1}_r$ and $\mathbf{2}_r$ may be eliminated so that by $\mathbf{Z}_1 - \mathbf{Z}_2 \cdot \mathbf{Z}_4^{-1} \cdot \mathbf{Z}_3$

$$Z' = \frac{1}{2}\;\begin{array}{c|c|c|} & \mathbf{1}_s & \mathbf{2}_s \\ \hline \mathbf{1}_s & r_s + jX_s + \dfrac{X_m^2}{\dfrac{r_r}{s} + jX_r} & 0 \\ \hline \mathbf{2}_s & 0 & r_s + jX_s + \dfrac{X_m^2}{\dfrac{r_r}{2-s} + jX_r} \\ \hline \end{array} = \frac{1}{2}\;\begin{array}{c|c|c|} & \mathbf{1}_s & \mathbf{2}_s \\ \hline \mathbf{1}_s & Z_1 & \\ \hline \mathbf{2}_s & & Z_2 \\ \hline \end{array} \qquad 21.13$$

where $s = 1 - v$ and $2 - s = 1 + v$. Also Z_1 = positive-sequence reactance and Z_2 = negative-sequence reactance.

Three-Phase Induction Motor with Unbalanced Voltages

(*a*) Let it be assumed that both stator and rotor of the induction motor are three-phase. Then along the **d**, **q**, and **0** axes the **Z**, **G**, and **e** tensors are the same as those of the two-phase motor, except that in **Z** two additional zero-sequence rows and columns are introduced with $Z_0 = r_0 + jX_0$.

$$\mathbf{Z} = \begin{array}{c|c|c|c|c|c|c|} & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s & \mathbf{0}_s & \mathbf{0}_r \\ \hline \mathbf{d}_s & r_s + jX_s & jX_m & & & & \\ \hline \mathbf{d}_r & jX_m & r_r + jX_r & X_r v & X_m v & & \\ \hline \mathbf{q}_r & -X_m v & -X_r v & r_r + jX_r & jX_m & & \\ \hline \mathbf{q}_s & & & jX_m & r_s + jX_s & & \\ \hline \mathbf{0}_s & & & & & r_{0r} + jX_{0s} & \\ \hline \mathbf{0}_r & & & & & & r_{0r} + jX_{0r} \\ \hline \end{array} \qquad 21.14$$

$$\mathbf{e} = \begin{array}{|c|c|c|c|c|c|} \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s & \mathbf{0}_s & \mathbf{0}_r \\ \hline e_{ds} & & & e_{qs} & e_0 & \\ \hline \end{array}$$

The **G** tensor remains the same as equation 21.5.

If the **d** and **q** axes are transformed to **1** and **2** (or rather if **d**, **q**, and **0** are transformed to **0**, **1**, and **2**), the **C** has the same form as before, equation 21.6.

$$\mathbf{C} = \frac{1}{2}\;\begin{array}{c|cccccc|} & \mathbf{0}_s & \mathbf{1}_s & \mathbf{2}_s & \mathbf{0}_r & \mathbf{1}_r & \mathbf{2}_r \\ \hline \mathbf{d}_s & & 1 & 1 & & & \\ \mathbf{d}_r & & & & & 1 & 1 \\ \mathbf{q}_r & & & & & -j & j \\ \mathbf{q}_s & & -j & j & & & \\ \mathbf{0}_s & 2 & & & & & \\ \mathbf{0}_r & & & & 2 & & \\ \hline \end{array} \qquad 21.15$$

(*b*) If the steps of the previous section are repeated, that is, if **Z′**, **G′**, and **e′** are calculated, the same results are found as before except that **Z** has an additional $\mathbf{0}_s$ axis. Leaving out $\frac{1}{2}$ in **Z′** (but not in **e′**) and eliminating also the rotor axes

$$\mathbf{Z}'' = \begin{array}{c|ccc|} & \mathbf{0}_s & \mathbf{1}_s & \mathbf{2}_s \\ \hline \mathbf{0}_s & Z_0 & & \\ \mathbf{1}_s & & Z_1 & \\ \mathbf{2}_s & & & Z_2 \\ \hline \end{array} \qquad \mathbf{e}' = \begin{array}{c|c|} \mathbf{0}_s & e_0 \\ \mathbf{1}_s & e_1 \\ \mathbf{2}_s & e_2 \\ \end{array} \qquad 21.16$$

where Z_1, Z_2, e_1, and e_2 are defined in equations 21.13 and 21.9 and $Z_0 = r_{0s} + jx_{0s}$. All constants r_s, X_s, X_m, X_r, and r_r are for one phase (line to neutral).

The currents are found by $\mathbf{i} = \mathbf{Z}^{-1}\cdot\mathbf{e}$ and the torque per phase by the real part of $\mathbf{i}^*\cdot\omega\mathbf{G}\cdot\mathbf{i}$.

(*c*) When a three-phase induction motor operates under unbalanced condition, it is necessary to express its performance in terms of sequence currents, since then the torque calculation is comparatively simple (**G″** has only two non-zero diagonal components). In any other reference frame **G** has nearly nine components. (For additional examples see *A.T.E.M.*, p. 59.)

The Capacitor Motor

(*a*) If the cross-phase turns are a times the main phase turns, then **C** has unity in all diagonal components, except a in axis $\mathbf{q}_s$, Table VI–5.

If an impedance Z is added to axis q_s (Fig. 21.3) (Z represents any dissymmetry in the impedances of the two stator windings, also any added condenser), Z' and G' of the resultant system are

$$\mathbf{C}_1 = \begin{array}{c|cccc|} & d_s & d_r & q_r & q_s \\ \hline d_s & 1 & & & \\ d_r & & 1 & & \\ q_r & & & 1 & \\ q_s & & & & a \\ \hline \end{array} \qquad \mathbf{Z}' = \begin{array}{c|cccc|} & d_s & d_r & q_r & q_s \\ \hline d_s & r_s + jX_s & jX_m & & \\ d_r & jX_m & r_r + jX_r & X_r v & aX_m v \\ q_r & -X_m v & -X_r v & r_r + jX_r & ajX_m \\ q_s & & & ajX_m & a^2(r_s + jX_s) + Z \\ \hline \end{array} \qquad 21.17$$

$$\omega\mathbf{G}' = \begin{array}{c|cccc|} & d_s & d_r & q_r & q_s \\ \hline d_s & & & & \\ d_r & & & X_r & aX_m \\ q_r & -X_m & -X_r & & \\ q_s & & & & \\ \hline \end{array} \qquad \mathbf{e}' = \begin{array}{c|c|} d_s & e_d \\ d_r & \\ q_r & \\ q_s & ae_q \\ \hline \end{array} \qquad 21.18$$

where $Z = R + jX$ and (if $r_s = r_{sd}$ and $x_s = x_{sd}$)

$$R = (r_{sq} - r_{sd}a^2) + R_c \qquad X = (x_{sq} - x_{sd}a^2) - X_c \qquad 21.19$$

so that

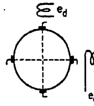


FIG. 21.3. Capacitor motor.

$$a^2(r_s + jX_m) + Z = r_{sq} + R_c + jX_{sq} - jX_c$$

The equations $\mathbf{e}' = \mathbf{Z}' \cdot \mathbf{i}'$ represent the cross-field theory of the capacitor motor.

(*b*) In order to introduce revolving fields in both stator and rotor, *it is first necessary to change back the* $\mathbf{Z}'$ *of the capacitor motor to that of the primitive machine by* $\mathbf{C}_1^{-1}$ *and then only to use the standard* $\mathbf{C}_2$ *of the revolving-field theory. Thereby the* $\mathbf{C}$ *changing from the cross-field to the revolving-field theory is* $\mathbf{C} = \mathbf{C}_1^{-1} \cdot \mathbf{C}_2$, where

$$\mathbf{C}_1^{-1} = \begin{array}{c|cccc|} & d_s & d_r & q_r & q_s \\ \hline d_s & 1 & & & \\ d_r & & 1 & & \\ q_r & & & 1 & \\ q_s & & & & 1/a \\ \hline \end{array} \qquad \mathbf{C}_2 = \tfrac{1}{2} \begin{array}{c|cccc|} & 1_s & 2_s & 1_r & 2_r \\ \hline d_s & 1 & 1 & & \\ d_r & & & 1 & 1 \\ q_r & & & -j & j \\ q_s & -j & j & & \\ \hline \end{array} \qquad 21.20$$

Hence the resultant $\mathbf{C} = \mathbf{C}_1^{-1}\cdot\mathbf{C}_2$ is

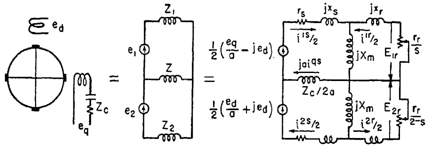
$$
\begin{aligned}
i^{ds} &= (i^{1s} + i^{2s})/2 \\
i^{dr} &= (i^{1r} + i^{2r})/2 \\
i^{qr} &= -j(i^{1r} - i^{2r})/2 \\
i^{qs} &= -j(i^{1s} - i^{2s})/2a
\end{aligned}
\qquad
\mathbf{C} = \mathbf{C}_1^{-1}\cdot\mathbf{C}_2 = \tfrac{1}{2}\;
\begin{array}{c|c|c|c|c|}
 & 1_s & 2_s & 1_r & 2_r \\
\hline
\mathbf{d}_s & 1 & 1 & & \\
\hline
\mathbf{d}_r & & & 1 & 1 \\
\hline
\mathbf{q}_r & & & -j & j \\
\hline
\mathbf{q}_s & -j/a & j/a & & \\
\hline
\end{array}
\qquad 21.21
$$

By $\mathbf{C}_t^*\cdot\mathbf{Z}\cdot\mathbf{C}$, etc.,

$$
\mathbf{Z}' = \tfrac{1}{2}\;
\begin{array}{c|c|c|c|c|}
 & 1_s & 2_s & 1_r & 2_r \\
\hline
1_s & r_s + jX_s + Z/4a^2 & -Z/4a^2 & jX_m & \\
\hline
2_s & -Z/4a^2 & r_s + jX_s + Z/4a^2 & & jX_m \\
\hline
1_r & jX_m & & \dfrac{r_r}{1-v} + jX_r & \\
\hline
2_r & & jX_m & & \dfrac{r_r}{1+v} + jX_r \\
\hline
\end{array}
\qquad 21.22
$$

$$
\mathbf{e}' = \;
\begin{array}{c|c|}
\hline
1_s & (e_d + je_q/a)/2 \\
\hline
2_s & (e_d - je_q/a)/2 \\
\hline
1_r & \\
\hline
2_r & \\
\hline
\end{array}
\quad 21.23
\qquad
\mathbf{E}' = \;
\begin{array}{c|c|}
\hline
1_s & \\
\hline
2_s & \\
\hline
1_r & -(jX_m i^{1s} + jX_r i^{1r})/2 \\
\hline
2_r & (jX_m i^{2s} + jX_r i^{2r})/2 \\
\hline
\end{array}
\quad 21.24
$$

(*c*) Since in $\mathbf{Z}'$ the reactance Z occurs in each component, both currents i^{1s} and i^{2s} must flow through it. Hence the equivalent circuit of the capacitor motor is that of Fig. 21.4*c*.



(*a*) (*b*) (*c*)

FIG. 21.4. Equivalent circuit of the capacitor (or split-phase) motor.

The current flowing through $Z/2a^2$ is jai^{qs}. The main phase current is $i^{ds} = (i^{1s} + i^{2s})/2$. The losses in r_r/s represent the positive sequence

torque per phase and those in $r_r(2 - s)$ the negative sequence torque per phase.

(*d*) Eliminating the rotor axes $\mathbf{1}_r$ and $\mathbf{2}_r$, just as in the unbalanced induction motor, then multiplying $\mathbf{Z}'$ by 2 (but not $\mathbf{e}'$), the result is

$$\mathbf{Z}' = \begin{array}{c|c|c|} & \mathbf{1}_s & \mathbf{2}_s \\ \hline \mathbf{1}_s & Z_1 + Z' & -Z' \\ \hline \mathbf{2}_s & -Z' & Z_2 + Z' \\ \hline \end{array} \qquad 21.25$$

where $Z' = Z/2a^2$, also Z_1 is the positive-sequence reactance and Z_2 is the negative-sequence reactance defined in equation 21.13.

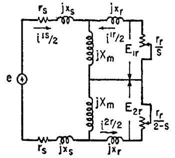


FIG. 21.5. Equivalent circuit of the single-phase induction motor.

(*e*) In special cases this equivalent circuit of the capacitor motor reduces to well-known circuits. In particular:

1. When $Z = 0$ and $a = 1$, the equivalent circuit becomes that of the balanced induction motor under unbalanced voltages, Fig. 21.2.

2. When $Z = \infty$, the circuit reduces to that of the standard single-phase induction motor, Fig. 21.5.

CHAPTER 22

POLYPHASE MACHINES*

Ignoring Half the Axes

(*a*) When the air gap is smooth and the windings along the **d** and **q** axes are identical, **Z** and **G** of the primitive machine are

$$
\mathbf{Z} = \begin{array}{c|cccc|}
 & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\
\hline
\mathbf{d}_s & r_s + L_s p & Mp & & \\
\mathbf{d}_r & Mp & r_r + L_r p & L_r p\theta & Mp\theta \\
\mathbf{q}_r & -Mp\theta & -L_r p\theta & r_r + L_r p & Mp \\
\mathbf{q}_s & & & Mp & r_s + L_s p \\
\hline
\end{array}
\qquad
\mathbf{G} = \begin{array}{c|cccc|}
 & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\
\hline
\mathbf{d}_s & & & & \\
\mathbf{d}_r & & & L_r & M \\
\mathbf{q}_r & -M & -L_r & & \\
\mathbf{q}_s & & & & \\
\hline
\end{array}
\qquad 22.1
$$

$$
\mathbf{e} = \begin{array}{|cccc|}
\mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\
\hline
e_{ds} & e_{dr} & e_{qr} & e_{qs} \\
\hline
\end{array}
$$

Much labor may be saved in the study of polyphase machines with smooth air gap by deriving their equations from that of a "primitive polyphase machine" containing only the windings of one of the phases, say those of the direct axis.

(*b*) Since all phenomena in the quadrature axes are identical to those in the direct axes, except that they take place 90 degrees later in time, at any instant $i^q = -ji^d$. Hence, for the above primitive machine, let the following transformation be introduced:

$$
\begin{aligned}
i^{ds} &= i^{ds'}/\sqrt{2} \\
i^{dr} &= i^{dr'}/\sqrt{2} \\
i^{qr} &= -ji^{dr'}/\sqrt{2} \\
i^{qs} &= -ji^{ds'}/\sqrt{2}
\end{aligned}
\qquad
\mathbf{C} = \frac{1}{\sqrt{2}}
\begin{array}{c|cc|}
 & \mathbf{d}_{s'} & \mathbf{d}_{r'} \\
\hline
\mathbf{d}_s & 1 & \\
\mathbf{d}_r & & 1 \\
\mathbf{q}_r & & -j \\
\mathbf{q}_s & -j & \\
\hline
\end{array}
\qquad 22.2
$$

* *A.T.E.M.*, p. 50.

Note that, except for the factor of $1/\sqrt{2}$, this transformation is identical to the positive-sequence portion of the method of two-phase symmetrical components, namely, equation 21.6.

(c) Another interpretation for this transformation may be given by finding the new voltage vector $\mathbf{e}'$

$$\mathbf{e}' = \mathbf{C}_t^* \cdot \mathbf{e} = \frac{1}{\sqrt{2}} \begin{array}{c|c|} & \\ \mathrm{d}_{s'} & e_{ds'} \\ \hline \mathrm{q}_{s'} & e_{qs'} \end{array} = \frac{1}{\sqrt{2}} \begin{array}{c|c|} & \\ \mathrm{d}_{s'} & e_{ds} + je_{qs} \\ \hline \mathrm{q}_{s'} & e_{dr} + je_{qr} \end{array} \qquad 22.3$$

This point of view states that, if the four currents and voltages in the *four* axes are *real* functions of time, they may be replaced by *two* cur-

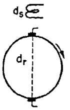


Fig. 22.1. The primitive polyphase machine with two layers of windings.

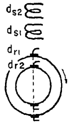


Fig. 22.2. The primitive polyphase machine with four layers of windings.

rents and voltages that are *complex* functions of time. Then the real parts of the new $\mathbf{e}'$ and $\mathbf{i}'$ give the direct axis quantities and the imaginary parts give the quadrature axis quantities.

Both points of view lead to the same set of equations.

The Primitive Polyphase Machine

(a) By $\mathbf{C}_t^* \cdot \mathbf{Z} \cdot \mathbf{C}$ and $\mathbf{C}_t^* \cdot \mathbf{G} \cdot \mathbf{C}$, equations 22.1 become (Fig. 22.1)

$$\mathbf{Z}' = \begin{array}{c|c|c|} & \mathrm{d}_s & \mathrm{d}_r \\ \hline \mathrm{d}_s & r_s + L_s p & Mp \\ \hline \mathrm{d}_r & M(p - jp\theta) & r_r + L_r(p - jp\theta) \\ \hline \end{array} \qquad \mathbf{G} = \begin{array}{c|c|c|} & \mathrm{d}_s & \mathrm{d}_r \\ \hline \mathrm{d}_s & & \\ \hline \mathrm{d}_r & -jM & -jL_r \\ \hline \end{array} \qquad 22.4$$

representing the $\mathbf{Z}'$ and $\mathbf{G}'$ tensors of the primitive polyphase machine with two layers. (Because of the smooth air gap, $-jL_r$ should be neglected in computations. In establishing equivalent circuits, however, $-jL_r$ must be included, so that $\mathbf{G}$ should be a tensor.)

(*b*) The results represent a theorem that a set of real equations of the form $\mathbf{e} = \mathbf{Z}\cdot\mathbf{i}$

$$\mathbf{e} = \begin{array}{c|c|} & \\ \mathrm{d} & e_d \\ \hline \mathrm{q} & e_q \end{array} \qquad \mathbf{Z} = \begin{array}{c|c|c|} & \mathrm{d} & \mathrm{q} \\ \mathrm{d} & r & -x \\ \hline \mathrm{q} & x & r \end{array} \qquad \mathbf{i} = \begin{array}{c|c|} & \\ \mathrm{d} & i^d \\ \hline \mathrm{q} & i^q \end{array} \tag{22.5}$$

may be replaced by the *complex* equation

$$\mathbf{e} = \begin{array}{|c|} \mathrm{d} \\ \hline e_d + je_q \end{array} \qquad \mathbf{Z} = \begin{array}{c|c|} & \mathrm{d} \\ \mathrm{d} & r + jX \end{array} \qquad \mathbf{i} = \begin{array}{|c|} \mathrm{d} \\ \hline i^d + ji^q \end{array} \tag{22.6}$$

and vice versa. (*A.T.E.M.*, p. 147.)

(*c*) By a similar transformation the Z of the primitive polyphase machine with *four layers* is (Fig. 22.2)

$$\mathbf{Z} = \begin{array}{c|c|c|c|c|} & \mathrm{d}_{s2} & \mathrm{d}_{s1} & \mathrm{d}_{r1} & \mathrm{d}_{r2} \\ \mathrm{d}_{s2} & r_{s2} + L_{s2}p & M_sp & M_{12}p & M_{22}p \\ \hline \mathrm{d}_{s1} & M_sp & r_{s1} + L_{s1}p & M_{11}p & M_{21}p \\ \hline \mathrm{d}_{r1} & M_{12}(p - jp\theta) & M_{11}(p - jp\theta) & r_{r1} + L_{r1}(p - jp\theta) & M_r(p - jp\theta) \\ \hline \mathrm{d}_{r2} & M_{22}(p - jp\theta) & M_{21}(p - jp\theta) & M_r(p - jp\theta) & r_{r2} + L_{r2}(p - jp\theta) \end{array} \tag{22.7}$$

$$\mathbf{G} = \begin{array}{c|c|c|c|c|} & \mathrm{d}_{s2} & \mathrm{d}_{s1} & \mathrm{d}_{r1} & \mathrm{d}_{r2} \\ \mathrm{d}_{s2} & & & & \\ \hline \mathrm{d}_{s1} & & & & \\ \hline \mathrm{d}_{r1} & -jM_{12} & -jM_{11} & -jL_{r1} & -jM_r \\ \hline \mathrm{d}_{r2} & -jM_{22} & -jM_{21} & -jM_{r2} & -jL_{r2} \end{array} \tag{22.8}$$

For *steady state*, when all axes have fundamental frequency currents in them, $p = j\omega$, $p - jp\theta = j\omega s = j\omega(1 - v)$.

Synchronous Machines

(*a*) When both stator and rotor axes rotate with the *rotor*, as in a synchronous machine, the equations are the same as when the axes stand still on the stator, except that axis **s** becomes **f** (field) and **r** becomes **a** (Fig. 22.3). The direction of rotation $p\theta$ also changes sign (see Fig. 16.11). Hence

$$
\mathbf{Z} = \begin{array}{c|c|c} & \mathbf{f} & \mathbf{a} \\ \hline \mathbf{f} & r_f + L_f p & Mp \\ \mathbf{a} & M(p + jp\theta) & r_a + L_a(p + jp\theta) \end{array}
\qquad
\mathrm{G}_a = \begin{array}{c|c|c} & \mathbf{f} & \mathbf{a} \\ \hline \mathbf{f} & & \\ \mathbf{a} & jM & jL_a \end{array}
\qquad
\mathrm{e} = \begin{array}{c|c} & \\ \hline \mathbf{f} & e_f \\ \mathbf{a} & e_a \end{array}
\qquad 22.9
$$

(*b*) When both i^f and i^a are assumed to be constant, then, in equation 22.9, $p = 0$. The first equation gives $i_f r_f = e_f$ or $i_f = e_f/r_f$. The second equation gives

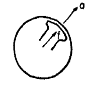


FIG. 22.3. Polyphase synchronous machine.

$$e_a - ji^f Mp\theta = (r_a + jL_a p\theta)i^a$$

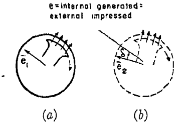
$$
\mathbf{Z} = \begin{array}{c|c} & \mathbf{a} \\ \hline \mathbf{a} & r_a + jL_a p\theta \end{array}
\qquad
\mathrm{e} = \begin{array}{c} \mathbf{a} \\ \hline e_a - ji^f Mp\theta \end{array}
\qquad 22.10
$$

That is, the excitation in the field appears as an impressed voltage $-ji^f Mp\theta$ in the armature (a vector along the negative q axis, as shown in Fig. 21.4*a*).

(*c*) When the synchronous machine is an infinite bus, its r_a and L_a are zero. Hence from equation 22.10

$$e_a = ji^f Mp\theta \qquad 22.11$$

This is the voltage impressed by an infinite bus upon the *rotating* axis of



(*a*) Alternator. (*b*) Infinite bus.

FIG. 22.4. Polyphase alternator connected to infinite bus.

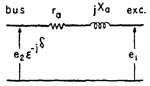


FIG. 22.5. Equivalent circuit of the polyphase alternator on infinite bus.

a polyphase machine (if its d axis is drawn along the field pole of the infinite bus).

(*d*) When an alternator is connected to an infinite bus (Fig. 22.4), the voltage impressed by the infinite bus along its own field axis q_2 is $\mathrm{e}_2 = ji^{f2}M_2p\theta_2$ (equation 22.11). As viewed from the alternator, e_2 lags behind e_1 by an angle δ. Hence during steady state

$$\mathbf{Z} = \begin{array}{c|c|} & \mathbf{a} \\ \hline \mathbf{a} & r_a + jX_a \\ \hline \end{array} \qquad \mathbf{e} = \begin{array}{|c|} \mathbf{a} \\ \hline e_2\epsilon^{-j\delta} - e_1 \\ \hline \end{array} \qquad 22.12$$

where $e_1 = ji^{f_1}M_1 p\theta_1$.

The equivalent circuit is given in Fig. 22.5. For a motor δ becomes negative.

Polyphase Machines with Unit Transformation Tensor

1. Sherbius Advancer (Fig. 22.6). Only the row and column of $\mathbf{d}_{r1}$ of equation 22.7 are used. When $p = j\omega$ and $p\theta = v\omega$,

$$\mathbf{Z} = \begin{array}{c|c|} & \mathbf{d}_r \\ \hline \mathbf{d}_r & r_r + jX_r(1 - v) \\ \hline \end{array} \qquad \mathbf{e} = \begin{array}{c|c|} \hline \mathbf{d}_r & \hat{e} \\ \hline \end{array}$$

$$\mathbf{G} = \begin{array}{c|c|} & \mathbf{d}_r \\ \hline \mathbf{d}_r & -jX_r \\ \hline \end{array} \qquad 22.13$$

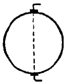


FIG. 22.6. Scherbius advancer.

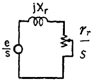


FIG. 22.7. Equivalent circuit of the Scherbius advancer.

When the rotor is above synchronism, $1 - v$ is negative and the *rotor acts as a condenser.*

Dividing $\mathbf{Z}$ and $\mathbf{e}$ by $1 - v = s$,

$$\mathbf{Z} = \begin{array}{c|c|} & \mathbf{d}_r \\ \hline \mathbf{d}_r & \frac{r_r}{s} + jX_r \\ \hline \end{array} \qquad \mathbf{e} = \begin{array}{c|c|} \hline \mathbf{d}_r & \frac{\hat{e}}{s} \\ \hline \end{array} \qquad 22.14$$

$$f = \text{Real of } \mathbf{i}^* \cdot \mathbf{E} = \mathbf{i} \cdot \mathbf{E}^*$$

The equivalent circuit is Fig. 22.7.

2. Double Squirrel-Cage Induction Motor (Fig. 22.8). The last three rows and columns of equation 22.7 are used. During steady state $p = j\omega$. Dividing the second and third row by s,

$$\mathbf{Z} = \begin{array}{c|c|c|c|} & \mathrm{s} & \mathrm{r}_1 & \mathrm{r}_2 \\ \hline \mathrm{s} & r_s + jX_s & jX_{m1} & jX_{m2} \\ \hline \mathrm{r}_1 & jX_{m1} & \frac{r_{r1}}{s} + jX_{r1} & jX_{mr} \\ \hline \mathrm{r}_2 & jX_{m2} & jX_{mr} & \frac{r_{r2}}{s} + jx_{r2} \\ \hline \end{array} \qquad \mathbf{G} = \begin{array}{c|c|c|c|} & \mathrm{s} & \mathrm{r}_1 & \mathrm{r}_2 \\ \hline \mathrm{s} & & & \\ \hline \mathrm{r}_1 & -jX_{m1} & -jX_r & -jX_{mr} \\ \hline \mathrm{r}_2 & -jX_{m2} & -jX_{mr} & -jX_{r2} \\ \hline \end{array} \qquad 22.15$$

Since $\mathbf{Z}$ is symmetrical, it may be represented by the equivalent sta-

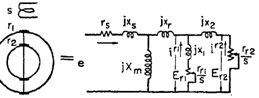


FIG. 22.8. Equivalent circuit of the double squirrel-cage induction motor.

tionary network with three meshes shown in Fig. 22.8, where $X_{m1} = X_{m2} = X_m$, and

$$X_s = X_m + x_s \qquad X_{r2} = X_m + x_r + x_2$$

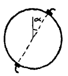
$$X_{r1} = X_m + x_r + x_1 \qquad X_{mr} = X_m + x_r$$

$$\mathbf{E} = \mathbf{G}\cdot\mathbf{i} = \begin{array}{c|c|} \mathrm{s} & \\ \hline \mathrm{r}_1 & E_{r1} \\ \hline \mathrm{r}_2 & E_{r2} \\ \hline \end{array}$$

$$f = \mathrm{Real}(i^{r1*}E_{r1} + i^{r2*}E_{r2}) = (i^{r1} + i^{r2})^*E_{r1}$$

The Shifting of Polyphase Brushes

When a set of perpendicular brushes is shifted by an angle α, Fig. 22.9, the first row of their $\mathbf{C}$ is



$$\mathbf{C} = \mathrm{d}_r \begin{array}{|c|c|} \mathrm{m} & \mathrm{n} \\ \hline \cos\alpha & -\sin\alpha \\ \hline \end{array} \qquad 22.16$$

FIG. 22.9. Shifting a polyphase brush.

Since $i^n = -ji^m$, $\mathbf{C}$ becomes

$$\mathbf{C} = \mathrm{d}_r \begin{array}{|c|} \mathrm{m} \\ \hline \cos\alpha + j\sin\alpha \\ \hline \end{array} = \mathrm{d}_r \begin{array}{|c|} \mathrm{m} \\ \hline \epsilon^{j\alpha} \\ \hline \end{array} \qquad \mathbf{C}_t^* = \mathrm{m} \begin{array}{|c|} \mathrm{d}_r \\ \hline \epsilon^{-j\alpha} \\ \hline \end{array} \qquad 22.17$$

Hence *in polyphase machines clockwise rotation of axes is represented by $\epsilon^{j\alpha}$ and a counterclockwise rotation by $\epsilon^{-j\alpha}$.*

Polyphase Commutator Machines

1. Shunt Polyphase Commutator Motor (Fig. 22.10)

$$\mathbf{C} = \begin{array}{c|cc|} & \mathrm{d}_s & \mathrm{a} \\ \hline \mathrm{d}_s & 1 & \\ \mathrm{d}_r & & \epsilon^{j\alpha} \\ \hline \end{array} \qquad \mathbf{Z}' = \begin{array}{c|cc|} & \mathrm{d}_s & \mathrm{a} \\ \hline \mathrm{d}_s & r_s + jX_s & jX_m\epsilon^{j\alpha} \\ \mathrm{a} & jsX_m\epsilon^{-j\alpha} & r_r + jsX_r \\ \hline \end{array} \qquad \mathbf{e} = \begin{array}{c|c|} \hline \mathrm{d}_s & e_s \\ \mathrm{a} & e_a \\ \hline \end{array} \qquad 22.18$$

The presence of $\epsilon^{j\alpha}$ makes $\mathbf{Z}$ asymmetrical. Transforming it, however, by $\mathbf{C}'$, the symmetrical $\mathbf{Z}''$ is

$$\mathbf{C}' = \begin{array}{c|cc|} & \mathrm{d}_s & \mathrm{d}_r \\ \hline \mathrm{d}_s & 1 & \\ \mathrm{a} & & \epsilon^{-j\alpha} \\ \hline \end{array} \qquad \mathbf{Z}'' = \begin{array}{c|cc|} & \mathrm{d}_s & \mathrm{d}_r \\ \hline \mathrm{d}_s & r_s + jX_s & jX_m \\ \mathrm{d}_r & jX_m & r_r/s + jX_r \\ \hline \end{array} \qquad \mathbf{e}'' = \begin{array}{c|c|} \hline \mathrm{d}_s & e_s \\ \mathrm{d}_r & \frac{e_a}{s}\epsilon^{j\alpha} \\ \hline \end{array} \qquad 22.19$$

FIG. 22.10. Shunt polyphase commutator motor.

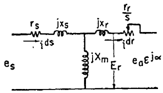


FIG. 22.11. Equivalent circuit of the shunt polyphase commutator motor.

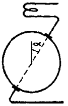


FIG. 22.12. Series polyphase commutator motor.

The equivalent circuit is that of Fig. 22.11. The machine i^a is found from i^{dr} of the equivalent circuit by $i^a = i^{dr}\epsilon^{-j\alpha}$.

$$\mathbf{G}'' = \begin{array}{c|cc|} & \mathrm{d}_s & \mathrm{d}_r \\ \hline \mathrm{d}_s & & \\ \mathrm{d}_r & -jX_m & -jX_r \\ \hline \end{array} \qquad \mathbf{E} = \begin{array}{c|c|} \hline \mathrm{d}_s & \\ \mathrm{d}_r & E_r \\ \hline \end{array} \qquad f = i^{dr*}E_r$$

2. Series Polyphase Commutator Motor (Fig. 22.12)

$$\mathbf{C}_1 = \begin{array}{c|cc|} & \mathrm{d}_s & \mathrm{a} \\ \hline \mathrm{d}_s & 1 & \\ \mathrm{d}_r & & \epsilon^{j\alpha} \\ \hline \end{array} \qquad \mathbf{C}_2 = \begin{array}{c|c|} & \mathrm{d}_s \\ \hline \mathrm{d}_s & n \\ \mathrm{a} & 1 \\ \hline \end{array} \qquad \mathbf{C} = \mathbf{C}_1 \cdot \mathbf{C}_2 = \begin{array}{c|c|} & \mathrm{d}_s \\ \hline \mathrm{d}_s & n \\ \mathrm{d}_r & \epsilon^{j\alpha} \\ \hline \end{array} \qquad 22.20$$

$$\mathbf{Z}' = \begin{array}{c|c|} & \mathrm{d}_s \\ \hline \mathrm{d}_s & \begin{array}{c} n^2(r_s + jX_s) + r_r \\ + jsX_r + jnX_m(\epsilon^{j\alpha} + s\epsilon^{-j\alpha}) \end{array} \\ \hline \end{array} \qquad \omega\mathbf{G}' = \begin{array}{c|c|} & \mathrm{d}_s \\ \hline \mathrm{d}_s & -jnX_m\epsilon^{-j\alpha} \\ \hline \end{array} \qquad 22.21$$

Its equivalent circuit is a variable impedance.

EXERCISES

1. Express the real equation $\mathbf{e} = \mathbf{Z}\cdot\mathbf{i}$ of the primitive machine with smooth air gap, given in equation 22.1, as a set of equations with complex coefficients, in the manner of equation 22.6.

2. Find C, Z, and G of the polyphase motor of Fig. 22.13. What is its torque?

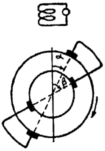


FIG. 22.13.

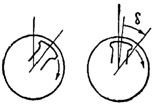


FIG. 22.14.

3. (*a*) Find Z, e, and **B** of two polyphase alternators in series, Fig. 22.14, the second lagging the first by a constant angle δ.

(*b*) Find the torque of each machine.

(*c*) Find the equivalent circuit of the system.

CHAPTER 23

ROTATING REFERENCE FRAMES*

C as a Function of Time

(*a*) Hitherto it has been assumed that the reference frames were (1) either all stationary in space (all fixed to the stator); (2) or rotating together with the same speed as one of the members (all fixed to the rotor).

The next point to investigate is how to establish the equations of a machine if the reference frames are not fixed to one member but rotate at any arbitrary velocity $p\theta'$. (The velocity of the rotor conductors will still be denoted by $p\theta$.)

(*b*) The first step is to establish **C** of a rotating frame. If the stationary axes **d** and **q** on Fig. 23.1 are to be replaced by the rotating

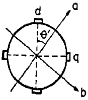


FIG. 23.1. Transformation from stationary to rotating axes.

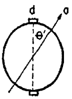


FIG. 23.2. Transformation of polyphase axes.

axes **a** and **b**, their **C** is analogous to the case where **a** and **b** are stationary (equation 17.4).

$$\begin{array}{cc} \mathbf{a} & \mathbf{b} \\ \hline \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{array} \qquad 23.1$$

except that now θ' *is a function of time and* $p\mathbf{C} = d\mathbf{C}/dt$ *is not zero.*

For a balanced polyphase machine in analogy to equation 22.17 (Fig. 23.2.)

$$\mathbf{C} = \begin{array}{c|c} & \mathbf{a} \\ \mathbf{d} & \epsilon^{j\theta'} \end{array} \qquad 23.2$$

(*c*) This is the first time when a **C** is encountered whose components are not constants (real or complex) but functions of time.

* *A.T.E.M.*, Parts VI and VII.

(*d*) Now, *when the components of* C *are functions of time, the laws of transformation of physical entities in general are more complicated than those hitherto shown.*

The Law of Transformation of Z

It will be proved presently that the law of transformation of Z is

$$\mathbf{Z}' = \mathbf{C}_t^* \cdot \mathbf{Z} \cdot \mathbf{C} + \mathbf{C}_t^* \cdot \mathbf{L} \cdot \frac{\partial \mathbf{C}}{\partial \theta'} p\theta' \quad \Big| \quad Z_{\alpha'\beta'} = Z_{\alpha\beta} C_{\alpha'}^{\alpha} C_{\beta'}^{\beta} + L_{\alpha\beta} C_{\alpha'}^{\alpha} \frac{\partial C_{\beta'}^{\beta}}{\partial \theta'} p\theta' \qquad 23.3$$

where C is a function of θ' and $p\theta'$ is the velocity of the reference frame. That is, now L (the coefficients of p terms) also have to be used. Because of this more complicated law of transformation, Z is no longer called a "tensor" but a "geometric object" (an entity whose existence depends on the reference frame used).

The law of transformation of all other tensors hitherto introduced, namely **i**, **e**, P, **R**, **L**, and G, are unchanged, and they are still called tensors, even though the reference frame rotates.

In rotating machinery it is often found that the transformation is "orthogonal," that is, $\mathbf{C}_t^* \cdot \mathbf{C}$ is the unit tensor. In such cases $\mathbf{C}_t^* \cdot \mathbf{Z} \cdot \mathbf{C}$ is often identical with Z and only the second term of equation 23.3 need be calculated.

A Quick Way of Transforming Z

When the transformation is not orthogonal, some labor may be saved by assuming that during the multiplication of Z with C *the order of the components is preserved.* Then it is possible to write for the law of transformation of Z

$$\mathbf{Z}' = \mathbf{C}_t^* \cdot \mathbf{Z} \cdot \mathbf{C} \quad \Big| \quad Z_{\alpha'\beta'} = Z_{\alpha\beta} C_{\alpha'}^{\alpha} C_{\beta'}^{\beta} \qquad 23.4$$

where *the p in* Z *now refers to all terms to the right of it, that is to* C (*and* **i**) *but not to* $\mathbf{C}_t^*$. *After multiplication each term may be expanded into two terms.* For instance, a component of $\mathbf{Z}' \cdot \mathbf{i}'$ may have the form $M \sin \theta' \, p \cos \theta' \, i^a$ (where $\cos \theta'$ came from C and $\sin \theta'$ from $\mathbf{C}_t^*$). Hence p refers to both terms $\cos \theta' \, i^a$. If the term is expanded, it becomes $M \sin \theta' \, p (\cos \theta' \, i^a) = M \sin \theta' \cos \theta' \, p i^a - M \sin^2 \theta' \, p\theta' \, i^a$. It is this last term that would have come from the use of $\mathbf{C}_t^* \cdot \mathbf{L} \cdot (\partial \mathbf{C}/\partial \theta') p\theta'$.

The Large Variety of Reference Frames Possible

In balanced polyphase machines it is advantageous to introduce reference frames rotating with the fluxes (or impressed voltages) since then all currents and fluxes become constant in magnitude and it is

possible to establish an equivalent circuit for the machine. In hunting studies the use of such a reference frame is imperative.

A great variety of reference frames is possible, their selection being influenced by the manner of interconnection of the machines. For instance, if both stator and rotor have rotating e.m.f.'s impressed on them, then:

1. Both stator and rotor reference axes may rotate with the stator e.m.f. (or flux).
2. Both may rotate with the rotor e.m.f.
3. The stator axis may rotate with the stator e.m.f. and the rotor axis with the rotor e.m.f.

Even though the two e.m.f.'s rotate at the same speed during steady state, still during hunting their speed is different and the equations of hunting depend upon where the reference axes are attached. A judicious selection of the reference frame may allow an easy solution of an otherwise prohibitively long problem.

Double-Fed Induction Motor

(*a*) In many speed-control systems the stator of an induction motor is connected to a synchronous machine running at a fundamental speed $p\theta_1$ and the slip rings are connected to another synchronous machine running at a slip speed $p\theta_3$. In that case the stator and rotor fluxes both run at a synchronous speed with a constant angle δ between them. Let both stator and rotor reference axes be attached to the revolving *stator* flux.

The tensors of the primitive machine of the induction motor with smooth air gap are

$$\mathbf{Z} = \begin{array}{c|c|c|c|c|} & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\ \hline \mathbf{d}_s & r_s + L_s p & Mp & & \\ \hline \mathbf{d}_r & Mp & r_r + L_r p & L_r p\theta_2 & Mp\theta_2 \\ \hline \mathbf{q}_r & -Mp\theta_2 & -L_r p\theta_2 & r_r + L_r p & Mp \\ \hline \mathbf{q}_s & & & Mp & r_s + L_s p \\ \hline \end{array} \qquad 23.5$$

$$\mathbf{L} = \begin{array}{c|c|c|c|c|} & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\ \hline \mathbf{d}_s & L_s & M & & \\ \hline \mathbf{d}_r & M & L_r & & \\ \hline \mathbf{q}_r & & & L_r & M \\ \hline \mathbf{q}_s & & & M & L_s \\ \hline \end{array} \qquad 23.6$$

$$\mathbf{e} = \begin{array}{c|c|}\cline{2-2} \mathrm{d}_s & -e_1 \sin \theta_1 \\ \cline{2-2} \mathrm{d}_r & -e_3 \sin (\theta_2 + \theta_3) \\ \cline{2-2} \mathrm{q}_r & e_3 \cos (\theta_2 + \theta_3) \\ \cline{2-2} \mathrm{q}_s & e_1 \cos \theta_1 \\ \cline{2-2} \end{array} \qquad 23.7$$

$$\mathbf{G} = \begin{array}{c|c|c|c|c|} & \mathrm{d}_s & \mathrm{d}_r & \mathrm{q}_r & \mathrm{q}_s \\ \cline{2-5} \mathrm{d}_s & & & & \\ \cline{2-5} \mathrm{d}_r & & & & M \\ \cline{2-5} \mathrm{q}_r & -M & & & \\ \cline{2-5} \mathrm{q}_s & & & & \\ \cline{2-5} \end{array} \qquad 23.8$$

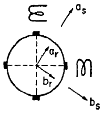


FIG. 23.3. Induction motor with rotating axes.

where $e_1 = i^{f_1} M_1 p\theta_1$ and $e_3 = i^{f_3} M_3 p\theta_3$ (compare with equation 24.16). The velocity of the rotor of the induction motor is $p\theta_2$; those of the synchronous machines, $p\theta_1$ and $p\theta_3$.

(*b*) Let a reference frame be introduced rotating with a velocity $p\theta_1$ with respect to the stationary reference axes. Then by equation 23.3.

$$\mathbf{C} = \begin{array}{c|c|c|c|c|} & \mathrm{a}_s & \mathrm{a}_r & \mathrm{b}_r & \mathrm{b}_s \\ \cline{2-5} \mathrm{d}_s & \cos \theta_1 & & & -\sin \theta_1 \\ \cline{2-5} \mathrm{d}_r & & \cos \theta_1 & -\sin \theta_1 & \\ \cline{2-5} \mathrm{q}_r & & \sin \theta_1 & \cos \theta_1 & \\ \cline{2-5} \mathrm{q}_s & \sin \theta_1 & & & \cos \theta_1 \\ \cline{2-5} \end{array} \qquad 23.9$$

$$\frac{\partial \mathbf{C}}{\partial \theta_1} = \begin{array}{|c|c|c|c|} \hline -\sin \theta_1 & & & -\cos \theta_1 \\ \hline & -\sin \theta_1 & -\cos \theta_1 & \\ \hline & \cos \theta_1 & -\sin \theta_1 & \\ \hline \cos \theta_1 & & & -\sin \theta_1 \\ \hline \end{array} \qquad 23.10$$

Now $\mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C}$ happens to assume the original form of $\mathbf{Z}$, equation 23.5. But

$$\mathbf{C}_t \cdot \mathbf{L} \cdot \frac{\partial \mathbf{C}}{\partial \theta_1} p\theta_1 = \mathbf{V}' p\theta_1 = \begin{array}{c|c|c|c|c|} & \mathrm{a}_s & \mathrm{a}_r & \mathrm{b}_s & \mathrm{b}_r \\ \cline{2-5} \mathrm{a}_s & & & -Mp\theta_1 & -L_s p\theta_1 \\ \cline{2-5} \mathrm{a}_r & & & -L_r p\theta_1 & -Mp\theta_1 \\ \cline{2-5} \mathrm{b}_r & Mp\theta_1 & L_r p\theta_1 & & \\ \cline{2-5} \mathrm{b}_s & L_s p\theta_1 & Mp\theta_1 & & \\ \cline{2-5} \end{array} \qquad 23.11$$

The sum of equations 23.5 and 23.11 is

$$\mathbf{Z}' = \begin{array}{c} \\ \mathbf{a}_s \\ \mathbf{a}_r \\ \mathbf{b}_r \\ \mathbf{b}_s \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{a}_s & \mathbf{a}_r & \mathbf{b}_r & \mathbf{b}_s \end{array} \\ \left[\begin{array}{c|c|c|c} r_s + L_s p & Mp & -Mp\theta_1 & -L_s p\theta_1 \\ \hline Mp & r_r + L_r p & -L_r p\theta_s & -Mp\theta_s \\ \hline Mp\theta_s & L_r p\theta_s & r_r + L_r p & Mp \\ \hline L_s p\theta_1 & Mp\theta_1 & Mp & r_s + L_s p \end{array}\right] \end{array} \qquad 23.12$$

$$\mathbf{G}' = \begin{array}{c} \\ \mathbf{a}_s \\ \mathbf{a}_r \\ \mathbf{b}_r \\ \mathbf{b}_s \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{a}_s & \mathbf{a}_r & \mathbf{b}_r & \mathbf{b}_s \end{array} \\ \left[\begin{array}{c|c|c|c} & & & \\ \hline & & & M \\ \hline -M & & & \\ \hline & & & \end{array}\right] \end{array} \qquad 23.13$$

where the slip speed is $p\theta_s = p\theta_1 - p\theta_2$. The torque is $M(i^{bs}i^{ar} - i^{as}i^{br})$ and

$$\mathbf{e}' = \mathbf{C}_t \cdot \mathbf{e} = \begin{array}{c} \begin{array}{cccc} \mathbf{a}_s & \mathbf{a}_r & \mathbf{b}_r & \mathbf{b}_s \end{array} \\ \left[\begin{array}{c|c|c|c} & -e_3 \sin \delta & e_3 \cos \delta & e_1 \end{array}\right] \end{array} \qquad 23.14$$

where $\delta = \theta_2 + \theta_3 - \theta_1$ = the angle between the two fluxes.

From the new axes **a** and **b** it appears that the rotor rotates with a slip velocity $p\theta_s$ and the stator with a fundamental velocity $p\theta_1$, *both in counterclockwise direction.*

(*c*) Since the impressed voltages are constant during steady state, all p in **Z** become zero and

$$\mathbf{Z}' = \begin{array}{c} \\ \mathbf{a}_s \\ \mathbf{a}_r \\ \mathbf{b}_r \\ \mathbf{b}_s \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{a}_s & \mathbf{a}_r & \mathbf{b}_r & \mathbf{b}_s \end{array} \\ \left[\begin{array}{c|c|c|c} r_s & & -X_m & -X_s \\ \hline & r_r & -sX_r & -sX_m \\ \hline sX_m & sX_r & r_r & \\ \hline X_s & X_m & & r_s \end{array}\right] \end{array} \qquad 23.15$$

$$\mathbf{G}' = \begin{array}{c|cccc} & \mathrm{a}_s & \mathrm{a}_r & \mathrm{b}_r & \mathrm{b}_s \\ \hline \mathrm{a}_s & & & & \\ \mathrm{a}_r & & & & X_m \\ \mathrm{b}_r & -X_m & & & \\ \mathrm{b}_s & & & & \end{array} \qquad 23.16$$

In $\mathbf{i}' = \mathbf{Z}'^{-1}\cdot\mathbf{e}'$, i^{as} is the in-phase component and i^{bs} is the out-of-phase component of the stator current.

EXERCISES

1. Find equation 23.12 from 23.5 with the aid of equation 23.4.

2. Transform equation 23.12 back to the original equation 23.5 with the aid of $\mathbf{C}^{-1}$.

3. Express C of equation 23.9 as a tensor with complex coefficients having only two rows and columns.

4. Transform Z of the primitive polyphase machine, equation 22.4, with C of exercise 3 by using equations 23.3 and 23.4. (The result should be the complex form of equation 23.12.)

CHAPTER 24

HOLONOMIC REFERENCE FRAMES

Axes Rotating with the Members

(*a*) A very important special case occurs when the axes are rigidly connected to their particular members and rotate with them. Such a reference frame may be assumed on the slip-ring induction motor and on the synchronous machine. In the synchronous machine the armature axes are then stationary and the field axes rotate with the field. Because many practical machines can be derived from it with the aid of a **C**, a machine with axes rigidly connected to the members will be called the "primitive machine with rotating axes" or the "second primitive machine."

When the reference frame is rigidly connected to the members (be they stationary or rotating) the equation of voltage reduces to the special case (to be proved presently)

$$\begin{array}{l|l} \mathbf{e}' = \mathbf{R}'\cdot\mathbf{i}' + p(\mathbf{L}'\cdot\mathbf{i}') & e_{\alpha'} = R_{\alpha'\beta'}i^{\beta'} + p(L_{\alpha'\beta'}i^{\beta'}) \\ \mathbf{e}' = \mathbf{R}'\cdot\mathbf{i}' + p\varphi' & e_{\alpha'} = R_{\alpha'\beta'}i^{\beta'} + p\varphi_{\alpha'} \end{array} \qquad 24.1$$

No rotor-generated voltage $\mathbf{B}p\theta$ exists (or rather none is defined) and the equation of voltage is the same as that of a stationary network. However, *p refers not only to* **i** *but also to* **L**, which now is a function of θ. When all expressions are expanded so that p refers only to **i**, the equations assume the usual form involving $\mathbf{G}p\theta\cdot\mathbf{i}$.

(*b*) The torque may be expressed as either

$$\begin{array}{l|l} f' = \mathbf{i}'\cdot\mathbf{G}'\cdot\mathbf{i}' & f' = G_{\alpha'\beta'}i^{\alpha'}i^{\beta'} \\ \text{or} \quad f' = \dfrac{\partial T'}{\partial\theta} = \dfrac{1}{2}\,\mathbf{i}'^{*}\cdot\dfrac{\partial\mathbf{L}'}{\partial\theta}\cdot\mathbf{i}' & f' = \dfrac{\partial T'}{\partial\theta} = \dfrac{1}{2}\dfrac{\partial L_{\alpha'\beta'}}{\partial\theta}\,i^{\alpha'}i^{\beta'} \end{array} \qquad 24.2$$

since the kinetic energy is

$$\begin{array}{l|l} T' = \frac{1}{2}\,\mathbf{i}'^{*}\cdot\mathbf{L}\cdot\mathbf{i}' & T' = \frac{1}{2}\,L_{\alpha'\beta'}i^{\alpha'}i^{\beta'} \end{array} \qquad 24.3$$

(*c*) These equations (valid for the special case of rigidly connected reference frames) are due to Maxwell. The reference axes are called "holonomic" axes. It is emphasized that these simplified equations are not valid for any other type of reference frame.

The Second Primitive Machine*

(*a*) Instead of transforming **Z** of the first primitive machine, it is simpler to transform its **L** by $\mathbf{C}_t \cdot \mathbf{L} \cdot \mathbf{C}$ to give $\mathbf{L}'$. Then

$$Z' = \mathbf{R}' + p\mathbf{L}' \quad \Big| \quad Z_{\alpha'\beta'} = R_{\alpha'\beta'} + pL_{\alpha'\beta'} \qquad 24.4$$

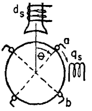


FIG. 24.1. Second primitive machine.

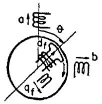


FIG. 24.2. Alternator with stationary armature axes.

($\mathbf{R}'$ is $\mathbf{C}_t \cdot \mathbf{R} \cdot \mathbf{C}$ and has the same form as **R**.) Hence

$$\mathbf{C} = \begin{array}{c|cccc} & d_s & a & b & q_s \\ \hline d_s & 1 & & & \\ d_r & & \cos\theta & -\sin\theta & \\ q_r & & \sin\theta & \cos\theta & \\ q_s & & & & 1 \end{array} \qquad \mathbf{L} = \begin{array}{c|cccc} & d_s & d_r & q_r & q_s \\ \hline d_s & L_{ds} & M_d & & \\ d_r & M_d & L_{dr} & & \\ q_r & & & L_{qr} & M_q \\ q_s & & & M_q & L_{qs} \end{array} \qquad 24.5$$

$$\mathbf{L}' = \begin{array}{c|cccc} & d_s & a & b & q_s \\ \hline d_s & L_{ds} & M_d\cos\theta & -M_d\sin\theta & 0 \\ a & M_d\cos\theta & L_S + L_D\cos 2\theta & -L_D\sin 2\theta & M_q\sin\theta \\ b & -M_d\sin\theta & -L_D\sin 2\theta & L_S - L_D\cos 2\theta & M_q\cos\theta \\ q_s & 0 & M_q\sin\theta & M_q\cos\theta & L_{qs} \end{array} \qquad 24.6$$

where $= L_S = \dfrac{L_{dr} + L_{qr}}{2}$ and $L_D = \dfrac{L_{dr} - L_{qr}}{2}$.

For the synchronous machine of Fig. 24.2 ($\mathbf{e}_g = \mathbf{Z}_g \cdot \mathbf{i}$)

$$\mathbf{L}_g = \begin{array}{c|cccc} & d_f & a & b & q_f \\ \hline d_f & -L_{df} & -M_d\cos\theta & -M_d\sin\theta & 0 \\ a & -M_d\cos\theta & -L_S - L_D\cos 2\theta & -L_D\sin 2\theta & M_q\sin\theta \\ b & -M_d\sin\theta & -L_D\sin 2\theta & -L_S + L_D\cos 2\theta & -M_q\cos\theta \\ q_f & 0 & M_q\sin\theta & -M_q\cos\theta & -L_{qs} \end{array} \qquad 24.7$$

* *A.T.E.M.*, p. 71.

The impedance tensor $\mathbf{R}' + p\mathbf{L}'$ is

$$Z' = \begin{array}{c|cccc} & \mathbf{d}_s & \mathbf{a} & \mathbf{b} & \mathbf{q}_s \\ \hline \mathbf{d}_s & r_{ds} + pL_{ds} & M_d p \cos\theta & -M_d p \sin\theta & 0 \\ \mathbf{a} & M_d p \cos\theta & r_r + p(L_S + L_D \cos 2\theta) & -L_D p \sin 2\theta & M_q p \sin\theta \\ \mathbf{b} & -M_d p \sin\theta & -L_D p \sin 2\theta & r_r + p(L_S - L_D \cos 2\theta) & M_q p \cos\theta \\ \mathbf{q}_s & 0 & M_q p \sin\theta & M_q p \cos\theta & r_{qs} + L_{qs} p \end{array} \qquad 24.8$$

where p refers to all θ terms and to i.

(*b*) To find the torque by $1/2\, \mathbf{i}' \cdot (\partial \mathbf{L}'/\partial\theta) \cdot \mathbf{i}'$

$$\frac{\partial \mathbf{L}'}{\partial \theta} = \begin{array}{c|cccc} & \mathbf{d}_s & \mathbf{a} & \mathbf{b} & \mathbf{q}_s \\ \hline \mathbf{d}_s & 0 & -M_d \sin\theta & -M_d \cos\theta & 0 \\ \mathbf{a} & -M_d \sin\theta & -2L_D \sin 2\theta & -2L_D \cos 2\theta & M_q \cos\theta \\ \mathbf{b} & -M_d \cos\theta & -2L_D \cos 2\theta & 2L_D \sin 2\theta & -M_q \sin\theta \\ \mathbf{q}_s & 0 & M_q \cos\theta & -M_q \sin\theta & 0 \end{array} \qquad 24.9$$

The torque may also be found by $\mathbf{i}' \cdot \mathbf{G}' \cdot \mathbf{i}'$ where $\mathbf{G}' = \mathbf{C}_t \cdot \mathbf{G} \cdot \mathbf{C} =$

$$\mathbf{G}' = \begin{array}{c|cccc} & \mathbf{d}_s & \mathbf{a} & \mathbf{b} & \mathbf{q}_s \\ \hline \mathbf{d}_s & & & & \\ \mathbf{a} & -M_d \sin\theta & -L_D \sin 2\theta & L_S - L_D \cos 2\theta & M_q \cos\theta \\ \mathbf{b} & -M_d \cos\theta & -(L_S + L_D \cos 2\theta) & L_D \sin 2\theta & -M_q \sin\theta \\ \mathbf{q}_s & & & & \end{array} \qquad 24.10$$

The Second Primitive Polyphase Machine*

When a machine with a smooth air gap has a pure rotating field on both stator and rotor, then (Fig. 24.3)

$$\mathbf{C} = \begin{array}{c|cc} & \mathbf{d}_s & \mathbf{a} \\ \hline \mathbf{d}_s & 1 & \\ \mathbf{d}_r & & \epsilon^{j\theta} \end{array} \qquad \mathbf{L} = \begin{array}{c|cc} & \mathbf{d}_s & \mathbf{a} \\ \hline \mathbf{d}_s & L_s & M \\ \mathbf{a} & M & L_r \end{array} \qquad 24.11$$

* *A.T.E.M.*, p. 74.

$$\mathbf{L}' = \begin{array}{c|c|c|} & \mathbf{d}_s & \mathbf{a} \\ \hline \mathbf{d}_s & L_s & M\epsilon^{j\theta} \\ \hline \mathbf{a} & M\epsilon^{-j\theta} & L_r \\ \hline \end{array} \qquad \frac{\partial \mathbf{L}'}{\partial \theta} = \begin{array}{c|c|c|} & \mathbf{d}_s & \mathbf{a} \\ \hline \mathbf{d}_s & & jM\epsilon^{j\theta} \\ \hline \mathbf{a} & -jM\epsilon^{-j\theta} & \\ \hline \end{array} \qquad 24.12$$

$$\mathbf{Z}' = \begin{array}{c|c|c|} & \mathbf{d}_s & \mathbf{a} \\ \hline \mathbf{d}_s & r_s + pL_s & pM\epsilon^{j\theta} \\ \hline \mathbf{a} & pM\epsilon^{-j\theta} & r_r + pL_r \\ \hline \end{array} \qquad \mathbf{G}' = \begin{array}{c|c|c|} & \mathbf{d}_s & \mathbf{a} \\ \hline \mathbf{d}_s & & \\ \hline \mathbf{a} & -jM\epsilon^{-j\theta} & \\ \hline \end{array}$$

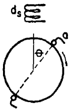


FIG. 24.3. Second polyphase primitive machine.

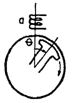


FIG. 24.4. Polyphase synchronous machine.

FIG. 24.5. Alternator with constant excitation.

This $\mathbf{Z}'$ and $\mathbf{G}'$ are the same as those of the slip-ring induction motor.

$$\text{Torque} = f' = \text{Real of } \mathbf{i}'^{*} \cdot \mathbf{G}' \cdot \mathbf{i}' \qquad 24.13$$

Polyphase Synchronous Machine with Constant Excitation

If the primitive machine is looked upon as an alternator with stationary armature axes (Fig. 24.4), then $\mathbf{d}_s$ becomes $\mathbf{a}$ (armature) and $\mathbf{a}$ becomes $\mathbf{f}$ (field). (There is no need now to interchange the two members.)

$$\mathbf{Z} = \begin{array}{c|c|c|} & \mathbf{a} & \mathbf{f} \\ \hline \mathbf{a} & r_a + pL_a & pM\epsilon^{j\theta} \\ \hline \mathbf{f} & pM\epsilon^{-j\theta} & r_f + pL_f \\ \hline \end{array} \qquad \mathbf{e}' = \begin{array}{c|c|} \hline \mathbf{a} & e_a \\ \hline \mathbf{f} & e_f \\ \hline \end{array} \qquad 24.14$$

(b) If the excitation i^f is assumed to be constant the first equation becomes independent of the second

$$e_a = (j\epsilon^{j\theta}\, p\theta\, M)i^f + (r_a + pL_a)i^a$$

Eliminating the field axis, the tensors along the *stationary* armature axis $\mathbf{a}$ are (Fig. 24.5)

$$\mathbf{Z} = \begin{array}{c|c|} & \mathbf{a} \\ \hline \mathbf{a} & r_a + pL_a \\ \hline \end{array} \qquad \mathbf{e} = \begin{array}{|c|} \mathbf{a} \\ \hline e_a - ji^f\, p\theta\, M\epsilon^{j\theta} \\ \hline \end{array} \qquad \mathbf{B} = \begin{array}{|c|} \mathbf{a} \\ \hline ji^f\, M\epsilon^{j\theta} \\ \hline \end{array} \qquad 24.15$$

(**B** is the coefficient of all $p\theta$ terms, carried over to the right-hand side of the equation $\mathbf{e} = \mathbf{Z} \cdot \mathbf{i}$.)

(*c*) If the synchronous machine acts as an infinite bus ($r_a = L_a = 0$) then the voltage impressed upon a machine with axis **a** connected to the infinite bus is

$$\mathbf{e} = \begin{array}{c} \mathbf{a} \\ \hline \left| \; j i^f \, p\theta \, M \epsilon^{j\theta} \right. \end{array} \qquad 24.16$$

EXERCISES

1. Show that the torques of the second primitive machine found by equations 24.9 and 24.10 are equal.

2. On the second primitive polyphase machine, Fig. 24.6, let four axes exist:

(*a*) The stationary stator axis $\mathbf{d}_s$.
(*b*) The rotor axis **a** with a velocity $p\theta$.
(*c*) The stator flux $\mathbf{f}_s$ rotating with $p\theta_s$ with respect to the stator axis $\mathbf{d}_s$.
(*d*) The rotor flux $\mathbf{f}_r$ rotating with $p\theta_r$ with respect to the rotor axis **a**.

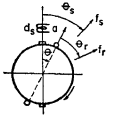


FIG. 24.6.

Find the following **C**'s and their inverse:

(*a*) From $\mathbf{d}_s$ to $\mathbf{f}_s$.
(*b*) From **a** to $\mathbf{f}_r$.
(*c*) From $\mathbf{d}_s$ to $\mathbf{f}_r$.
(*d*) From $\mathbf{f}_s$ to $\mathbf{f}_r$.

CHAPTER 25

SPEED CONTROL SYSTEMS

Changing Rotating Axes to Stationary Axes

Induction motors and synchronous machines are used in conjunction with a-c. commutator machines to produce desired speed and power factor characteristics for the drive of industrial loads. If each machine is a *balanced polyphase machine,* in the presence of slip-ring induction motors **Z**′ and **G**′ contain $\epsilon^{j\theta}$ terms. Such terms may be eliminated if *after interconnection* the slip-ring axes are replaced by stationary axes or if all axes are assumed to rotate with the fluxes.

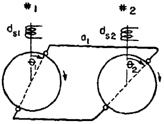
If two or more of the machines run in synchronism, then, after elimination of $\epsilon^{j\theta}$, usually their difference $\epsilon^{j\theta_1} - \epsilon^{j\theta_2} = \epsilon^{j\delta}$ remains, containing the constant angular displacement δ between the machines.

To establish an equivalent circuit for the polyphase system it is desirable that:

1. All reference axes should rotate together (no variable angle θ should exist between them).
2. All axes should be parallel (no constant angle δ should exist between them).

Power-Selsyn Systems

(*a*) Let two induction motors be interconnected as shown in Fig. 25.1. When machine 2 (transmitter) is driven, the other (receiver) runs at the same constant speed with a constant angle of lag δ.



Receiver. Transmitter.

FIG. 25.1. Selsyn system.

$$\mathbf{C}_1 = \begin{array}{c} \\ d_{s1} \\ a_1 \\ d_{s2} \\ a_2 \end{array} \begin{array}{c} \begin{array}{ccc} d_{s1} & a_1 & d_{s2} \end{array} \\ \left[\begin{array}{ccc} 1 & & \\ & 1 & \\ & & 1 \\ & -1 & \end{array}\right] \end{array} \qquad 25.1$$

Before interconnection

$$\mathbf{Z}' = \begin{array}{c} \\ \mathbf{d}_{s1} \\ \mathbf{a}_1 \\ \mathbf{d}_{s2} \\ \mathbf{a}_2 \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{d}_{s1} & \mathbf{a}_1 & \mathbf{d}_{s2} & \mathbf{a}_2 \end{array} \\ \left|\begin{array}{cccc} r_{s1} + L_{s1}p & M_1 p \epsilon^{j\theta_1} & & \\ M_1 p \epsilon^{-j\theta_1} & r_{r1} + L_{r1}p & & \\ & & r_{s2} + L_{s2}p & M_2 p \epsilon^{j\theta_2} \\ & & M_2 p \epsilon^{-j\theta_2} & r_{r2} + L_{r2}p \end{array}\right| \end{array} \qquad 25.2$$

After interconnection by $\mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C}$

$$\mathbf{Z}' = \begin{array}{c} \\ \mathbf{d}_{s1} \\ \mathbf{a}_1 \\ \mathbf{d}_{s2} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{d}_{s1} & \mathbf{a}_1 & \mathbf{d}_{s2} \end{array} \\ \left|\begin{array}{ccc} r_{s1} + L_{s1}p & M_1 p \epsilon^{j\theta_1} & 0 \\ M_1 p \epsilon^{-j\theta_1} & r_{r1} + r_{r2} + (L_{r1} + L_{r2})p & -M_2 p \epsilon^{-j\theta_2} \\ 0 & -M_2 p \epsilon^{j\theta_2} & r_{s2} + L_{s2}p \end{array}\right| \end{array} \qquad 25.3$$

Transforming from the rotating axis $\mathbf{a}_1$ to the stationary axis $\mathbf{d}_{r1}$ (Fig. 25.2),

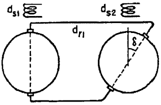


FIG. 25.2. Selsyn with stationary rotor axes.

$$\mathbf{C}_2 = \begin{array}{c} \\ \mathbf{d}_{s1} \\ \mathbf{a}_1 \\ \mathbf{d}_{s2} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{d}_{s1} & \mathbf{d}_{r1} & \mathbf{d}_{s2} \end{array} \\ \left|\begin{array}{ccc} 1 & & \\ & \epsilon^{-j\theta_1} & \\ & & 1 \end{array}\right| \end{array} \qquad 25.4$$

by $\mathbf{C}_t^* \cdot \mathbf{Z} \cdot \mathbf{C}$ (p in $\mathbf{Z}$ referring to $\mathbf{C}$ but not to $\mathbf{C}_t^*$)

$$\mathbf{Z}' = \begin{array}{c} \\ \mathbf{d}_{s1} \\ \mathbf{d}_{r1} \\ \mathbf{d}_{s2} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{d}_{s1} & \mathbf{d}_{r1} & \mathbf{d}_{s2} \end{array} \\ \left|\begin{array}{ccc} r_{s1} + L_{s1}p & M_1 p & 0 \\ \epsilon^{j\theta_1} M_1 p \epsilon^{-j\theta_1} & r_{r1} + r_{r2} + (L_{r1} + L_{r2}) \epsilon^{j\theta_1} p \epsilon^{-j\theta_1} & -M_2 \epsilon^{j\theta_1} p \epsilon^{-j\theta_2} \\ 0 & -M_2 p \epsilon^{j\theta_2} \epsilon^{-j\theta_1} & r_{s2} + L_{s2}p \end{array}\right| \end{array} \qquad 25.5$$

But $p(\epsilon^{-j\theta} i) = \epsilon^{-j\theta}(p - jp\theta)i$. Also $\theta_2 - \theta_1 = \delta$. Hence after expansion

$$\mathbf{Z}' = \begin{array}{c} \\ \mathbf{d}_{s1} \\ \mathbf{d}_{r1} \\ \mathbf{d}_{s2} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{d}_{s1} & \mathbf{d}_{r1} & \mathbf{d}_{s2} \end{array} \\ \left|\begin{array}{ccc} r_{s1} + L_{s1}p & M_1 p & 0 \\ M_1(p - jp\theta_1) & r_{r1} + r_{r2} + (L_{r1} + L_{r2})(p - jp\theta_1) & -M_2 \epsilon^{-j\delta}(p - jp\theta_2) \\ 0 & -M_2 p \epsilon^{j\delta} & r_{s2} + L_{s2}p \end{array}\right| \end{array} \qquad 25.6$$

(*b*) Since in every axis fundamental frequency currents flow, $p = j\omega$, $p - jp\theta = j\omega(1 - v) = j\omega s$. As no voltage is impressed in axis $\mathbf{d}_{r1}$, the whole row of $\mathbf{d}_{r1}$ may be divided by s. Hence during steady state

$$\mathbf{Z}' = \begin{array}{c|c|c|c|} & \mathbf{d}_{s1} & \mathbf{d}_{r1} & \mathbf{d}_{s2} \\ \hline \mathbf{d}_{s1} & r_{s1} + jX_{s1} & jX_{m1} & 0 \\ \hline \mathbf{d}_{r1} & jX_{m1} & (r_{r1} + r_{r2})/s + j(X_{r1} + X_{r2}) & -jX_{m2}\epsilon^{-j\delta} \\ \hline \mathbf{d}_{s2} & 0 & -jX_{m2}\epsilon^{j\delta} & r_{s2} + jX_{s2} \\ \hline \end{array} \qquad \mathbf{e} = \begin{array}{|c|} \hline \hat{e}_{s1} \\ \hline \\ \hline \hat{e}_{s2} \\ \hline \end{array} \qquad 25.7$$

Equivalent Circuit of the Selsyn System

The $\mathbf{Z}'$ in equation 25.7 may be brought to a diagonal form by shifting the axis of $\mathbf{d}_{s2}$ by the constant angle δ (Fig. 25.3).

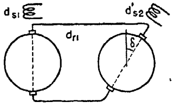


FIG. 25.3. Shifting the stator axes.

$$\mathbf{C}_3 = \begin{array}{c|c|c|c|} & \mathbf{d}_{s1} & \mathbf{d}_{r1} & \mathbf{d}_{s2} \\ \hline \mathbf{d}_{s1} & 1 & & \\ \hline \mathbf{d}_{r1} & & 1 & \\ \hline \mathbf{d}_{s2} & & & \epsilon^{j\delta} \\ \hline \end{array} \qquad 25.8$$

so that by $\mathbf{C}_{3t}^{*}\cdot\mathbf{Z}'\cdot\mathbf{C}_3$ the *symmetrical* $\mathbf{Z}''$ is

$$\mathbf{Z}'' = \begin{array}{c|c|c|c|} & \mathbf{d}_{s1} & \mathbf{d}_{r1} & \mathbf{d}'_{s2} \\ \hline \mathbf{d}_{s1} & r_{s1} + jX_{s1} & jX_{m1} & 0 \\ \hline \mathbf{d}_{r1} & jX_{m1} & (r_{r1} + r_{r2})/s + j(X_{r1} + X_{r2}) & -jX_{m2} \\ \hline \mathbf{d}'_{s2} & 0 & -jX_{m2} & r_{s2} + jX_{s2} \\ \hline \end{array} \qquad \mathbf{e}' = \begin{array}{c|c|} \hline \mathbf{d}_{s1} & \hat{e}_{s1} \\ \hline \mathbf{d}_{r1} & \\ \hline \mathbf{d}'_{s2} & \hat{e}_{s2}\epsilon^{-j\delta} \\ \hline \end{array} \qquad 25.9$$

The torque tensors before and after transformation are

$$\omega\mathbf{G}_1 = \begin{array}{c|c|c|} & \mathbf{d}_{s1} & \mathbf{a}_1 \\ \hline \mathbf{d}_{s1} & & \\ \hline \mathbf{a}_1 & -jX_{m1}\epsilon^{-j\theta_1} & -jX_{r1} \\ \hline \end{array} \qquad \omega\mathbf{G}_1'' = \begin{array}{c|c|c|} & \mathbf{d}_{s1} & \mathbf{d}_{r1} \\ \hline \mathbf{d}_{s1} & & \\ \hline \mathbf{d}_{r1} & -jX_{m1} & -jX_{r1} \\ \hline \end{array} \qquad 25.10$$

$$\omega\mathbf{G}_2 = \begin{array}{c|c|c|} & \mathbf{d}_{s2} & \mathbf{a}_2 \\ \hline \mathbf{d}_{s2} & & \\ \hline \mathbf{a}_2 & -jX_{m2}\epsilon^{-j\theta_2} & -jX_{r2} \\ \hline \end{array} \qquad \omega\mathbf{G}_2'' = \begin{array}{c|c|c|} & \mathbf{d}_{s2} & \mathbf{d}_{r1} \\ \hline \mathbf{d}_{s2} & & \\ \hline \mathbf{d}_{r1} & jX_{m2} & -jX_{r2} \\ \hline \end{array}$$

The torques are the real parts of

$$f_1 = i^{r1*}E_{r1} \qquad f_2 = i^{r1*}E_{r2}$$

The corresponding equivalent circuit is shown in Fig. 25.4. The actual current i^{s2} in the machine is $i^{s2} = i^{s2'}\epsilon^{j\delta}$. The same result

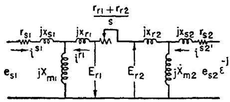


FIG. 25.4. Equivalent circuit of the Selsyn system.

would have been found by performing the three transformations in one step by $\mathbf{C} = \mathbf{C}_1 \cdot \mathbf{C}_2 \cdot \mathbf{C}_3$.

Variable-Speed Drive

(*a*) In a variable-speed drive two synchronous machines (the first supplying the electrical power) and a slip-ring induction motor (driving the load) are connected as shown in Fig. 25.5.

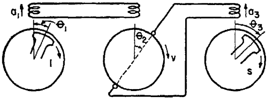


FIG. 25.5. Fan drive.

If during steady state the speed of the first machine is 1 and that of the induction motor is v, then the synchronous motor speed is $1 - v = s$. There is a constant angular displacement δ between the two induction motor fluxes that run with speeds of 1 and $v + s$.

Before interconnection the *transient* tensors are (equations 24.12 and 24.15)

$$\mathbf{Z} = \begin{array}{c|c|c|c|c|} & \mathrm{a}_1 & \mathrm{s}_2 & \mathrm{r}_2 & \mathrm{a}_3 \\ \hline \mathrm{a}_1 & r_1 + L_1p & & & \\ \hline \mathrm{s}_2 & & r_{s2} + L_{s2}p & pM_2\epsilon^{j\theta_2} & \\ \hline \mathrm{r}_2 & & pM_2\epsilon^{-j\theta_2} & r_{r2} + L_{r2}p & \\ \hline \mathrm{a}_3 & & & & r_3 + L_3p \\ \hline \end{array} \qquad \mathbf{e} = \begin{array}{c|c|} \mathrm{a}_1 & -ji^{f_1}M_1p\theta_1\epsilon^{j\theta_1} \\ \hline \mathrm{s}_2 & \\ \hline \mathrm{r}_2 & \\ \hline \mathrm{a}_3 & -ji^{f_3}M_3p\theta_3\epsilon^{j\theta_3} \\ \hline \end{array} \tag{25.11}$$

$$\mathbf{B}_1 = \mathbf{a}_1 \begin{bmatrix} ji^{f_1}M_1\epsilon^{j\theta_1} \end{bmatrix} \qquad \mathbf{G}_2 = \begin{array}{c} \\ \mathbf{s}_2 \\ \mathbf{r}_2 \end{array} \!\!\begin{array}{cc} \mathbf{s}_2 & \mathbf{r}_2 \\ \multicolumn{2}{c}{\begin{bmatrix} & \\ -jM_2\epsilon^{-j\theta_2} & \end{bmatrix}} \end{array} \qquad \mathbf{B}_3 = \mathbf{a}_3 \begin{bmatrix} ji^{f_3}M_3\epsilon^{j\theta_3} \end{bmatrix}$$

(*b*) The interconnection of the three machines is represented by

$$\mathbf{C}_1 = \begin{array}{c|cc} & \mathbf{s}_2 & \mathbf{r}_2 \\ \hline \mathbf{a}_1 & 1 & \\ \mathbf{s}_2 & -1 & \\ \mathbf{r}_2 & & -1 \\ \mathbf{a}_3 & & 1 \end{array} \tag{25.12}$$

When viewed from inside the induction motor, the reference axis $\mathbf{s}_2$ stands still on the stator (velocity 0) and $\mathbf{r}_2$ rotates with the rotor

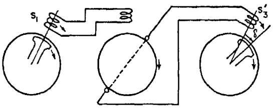


FIG. 25.6. Rotating armature axes.

(velocity $p\theta_2$). *Let both reference axes rotate with the rotor flux*, which rotates with respect to the rotor with a velocity $p\theta_3$. That is (Fig. 25.6), let

$$\mathbf{C}_2 = \begin{array}{c|cc} & \mathbf{s} & \mathbf{r} \\ \hline \mathbf{s}_2 & \epsilon^{j(\theta_2+\theta_3)} & \\ \mathbf{r}_2 & & \epsilon^{j\theta_3} \end{array} \tag{25.13}$$

The resultant $\mathbf{C}$ is

$$\mathbf{C} = \mathbf{C}_1 \cdot \mathbf{C}_2 = \begin{array}{c|cc} & \mathbf{s} & \mathbf{r} \\ \hline \mathbf{a}_1 & \epsilon^{j(\theta_2+\theta_3)} & \\ \mathbf{s}_2 & -\epsilon^{j(\theta_2+\theta_3)} & \\ \mathbf{r}_2 & & -\epsilon^{j\theta_3} \\ \mathbf{a}_3 & & \epsilon^{j\theta_3} \end{array} \tag{25.14}$$

(*c*) By $\mathbf{C}_t^* \cdot \mathbf{Z} \cdot \mathbf{C}$ (where p refers to $\mathbf{C}$ but not to $\mathbf{C}_t^*$) and by $\mathbf{C}_t^* \cdot \mathbf{e}$, $\mathbf{C}_t^* \cdot \mathbf{B}$, etc.,

$$\mathbf{Z}' = \begin{array}{c|c|c|} & \mathbf{s} & \mathbf{r} \\ \hline \mathbf{s} & r_1 + r_{s2} + (L_1 + L_{s2})[p + j(p\theta_2 + p\theta_3)] & M_2[p + j(p\theta_2 + p\theta_3)] \\ \hline \mathbf{r} & M_2(p + jp\theta_3) & r_{r2} + r_3 + (L_{r2} + L_3)(p + jp\theta_3) \\ \hline \end{array} \qquad 25.15$$

$$\mathbf{e}' = \begin{array}{c|c|} \hline \mathbf{s} & -\epsilon^{-j(\theta_2+\theta_3-\theta_1)} j i^{f_1} M_1 p\theta_1 \\ \hline \mathbf{r} & -j i^{f_3} M_3 p\theta_3 \\ \hline \end{array} \qquad \mathbf{B}'_1 = \begin{array}{c|c|} \hline \mathbf{s} & \epsilon^{-j(\theta_2+\theta_3-\theta_1)} j i^{f_1} M_1 \\ \hline \end{array} \qquad \mathbf{B}'_3 = \begin{array}{c|c|} \hline \mathbf{r} & j i^{f_3} M_3 \\ \hline \end{array}$$

$$\mathbf{G}'_2 = \begin{array}{c|c|c|} & \mathbf{s} & \mathbf{r} \\ \hline \mathbf{s} & & \\ \hline \mathbf{r} & -jM & -jXr \\ \hline \end{array}$$

(*d*) During steady state all currents are constant (as viewed from the frame), $p = 0$, also $p\theta_2 + p\theta_3 = \omega$ and $p\theta_3 = s\omega$. If $\theta_2 + \theta_3 - \theta_1 = \delta$ is the constant angle between the stator and rotor fluxes in the induction motor and if the axis $\mathbf{r}$ is divided by the slip s,

$$\mathbf{Z}' = \begin{array}{c|c|c|} & \mathbf{s} & \mathbf{r} \\ \hline \mathbf{s} & r_1 + r_{s2} + j(X_1 + X_{s2}) & jX_{m2} \\ \hline \mathbf{r} & jX_{m2} & \dfrac{r_{r2} + r_3}{s} + j(X_{r2} + X_3) \\ \hline \end{array} \qquad \mathbf{e}' = \begin{array}{c|c|} \hline \mathbf{s} & e_1\epsilon^{-j\delta} \\ \hline \mathbf{r} & e_3 \\ \hline \end{array} \qquad 25.16$$

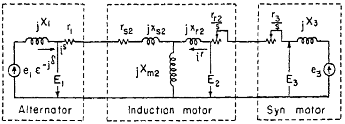


FIG. 25.7. Equivalent circuit of the fan drive.

where $\mathbf{e}_1 = -j i^{f_1} X_{m1} p\theta_1$. The equivalent circuit of the system is shown in Fig. 25.7. The torques are the real parts of

$$f_1 = i^{s*} E_1 \qquad f_2 = i^{r*} E_2 \qquad f_3 = i^{r*} E_3 \qquad 25.17$$

The steady-state equations and the equivalent circuit would have been the same (except for $\mathbf{e}'$) if both reference axes had rotated with the stator flux. In that case

$$\mathbf{C}_2 = \begin{array}{c|c|c|} & \mathbf{s}' & \mathbf{r}' \\ \hline \mathbf{s}_2 & \epsilon^{j\theta_1} & \\ \hline \mathbf{r}_r & & \epsilon^{j(\theta_1-\theta_2)} \\ \hline \end{array} \qquad \mathbf{e}' = \begin{array}{c|c|} \mathbf{s}' & e_1 \\ \hline \mathbf{r}' & e_3\epsilon^{j\delta} \\ \hline \end{array} \qquad 25.18$$

EXERCISE

1. Find $\mathbf{C}$, $\mathbf{Z}$, $\mathbf{i}$, and the torque of each machine of the following drives (all polyphase machines), Figs. 25.8–25.11.

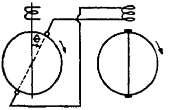


FIG. 25.8. Two induction motors in cascade.

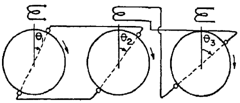


FIG. 25.9. Differential Selsyns.

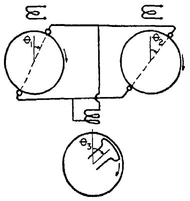


FIG. 25.10. Variable-speed drive.

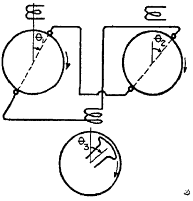


FIG. 25.11. Variable-speed drive.

CHAPTER 26

DERIVATION OF THE EQUATIONS FOR GENERAL ROTATING AXES

The Relative Concepts of Induced and Generated Voltages*

Let the first primitive machine with stationary axes be considered. In changing $\mathbf{i}$ to $\mathbf{i}'$ by $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$ to axes rotating with any *arbitrary* speed, let it again be assumed that the power input is the same when measured in either the rotating or in the stationary frames. That is, *let it be assumed that the power input is invariant under the transformation.* Then by equations 6.1 and 6.2 $\mathbf{e}$ is transformed as $\mathbf{e} = \mathbf{C}_t^{-1}\cdot\mathbf{e}'$.

(*a*) The part of $\mathbf{e} = \mathbf{Z}\cdot\mathbf{i}$ that contains $p = d/dt$ is $\mathbf{e}_i = \mathbf{L}\cdot p\mathbf{i}$, the induced voltage. Let the method of transforming the induced voltage to a frame rotating with $p\theta'$ be investigated.

Given: $\quad \mathbf{e} = \mathbf{L}\cdot\dfrac{d\mathbf{i}}{dt} \qquad\qquad e_\alpha = L_{\alpha\beta}\dfrac{di^\beta}{dt} \qquad 26.1$

Let $\quad \mathbf{i} = \mathbf{C}\cdot\mathbf{i}' \qquad\qquad i^\beta = C^\beta_{\beta'}i^{\beta'} \qquad 26.2$

and

$\quad \mathbf{e} = \mathbf{C}_t^{-1}\cdot\mathbf{e}' \qquad\qquad e_\alpha = C^{\alpha'}_\alpha e_{\alpha'} \qquad 26.3$

where $\mathbf{C}$ is a function of θ'. Substituting $\mathbf{i}$ and $\mathbf{e}$ into equation 26.1,

$$\mathbf{C}_t^{-1}\cdot\mathbf{e}' = \mathbf{L}\cdot\frac{d(\mathbf{C}\cdot\mathbf{i}')}{dt} \qquad\qquad C^{\alpha'}_\alpha e_{\alpha'} = L_{\alpha\beta}\frac{d(C^\beta_{\beta'}i^{\beta'})}{dt}$$

$$= \mathbf{L}\cdot\left(\mathbf{C}\cdot\frac{d\mathbf{i}'}{dt} + \frac{d\mathbf{C}}{dt}\cdot\mathbf{i}'\right) \qquad\qquad = L_{\alpha\beta}\left(C^\beta_{\beta'}\frac{di^{\beta'}}{dt} + \frac{dC^\beta_{\beta'}}{dt}i^{\beta'}\right)$$

Since $\mathbf{C}$ is a function of θ',

$$\frac{d\mathbf{C}}{dt} = \frac{\partial C}{\partial\theta'}\frac{d\theta'}{dt} = \frac{\partial C}{\partial\theta'}p\theta' \qquad\qquad \frac{dC^\beta_{\beta'}}{dt} = \frac{\partial C^\beta_{\beta'}}{\partial x^{\gamma'}}\frac{dx^{\gamma'}}{dt} = \frac{\partial C^\beta_{\beta'}}{\partial x^{\gamma'}}px^{\gamma'} \qquad 26.4$$

(As $\partial C^\alpha_{\alpha'}/\partial x^{\gamma'}$ is an object of valence 3 in every frame, in direct notation its product with other tensors cannot be represented in an easy

* *A.T.E.M.*, p. 61.

manner. Hence in direct notation only one velocity $p\theta' = v'$ is assumed; in index notation, any number $px^{\alpha'} = v^{\alpha'}$.) Substituting

$$\mathbf{C}_t^{-1}\cdot\mathbf{e}' = \mathbf{L}\cdot\left(\mathbf{C}\cdot\frac{d\mathbf{i}'}{dt} + \frac{\partial\mathbf{C}}{\partial\theta'}\cdot\mathbf{i}'p\theta'\right) \quad \Big| \quad C_\alpha^{\alpha'}e_{\alpha'} = L_{\alpha\beta}\left(C_{\beta'}^{\beta}\frac{di^{\beta'}}{dt}\frac{\partial C_{\beta'}^{\beta}}{\partial x^{\gamma'}}px^{\gamma'}i^{\beta'}\right)$$

Multiplying by $\mathbf{C}_t = C_{\alpha'}^{\alpha}$

$$\mathbf{e}' = \mathbf{C}_t\cdot\mathbf{L}\cdot\mathbf{C}\cdot\frac{d\mathbf{i}'}{dt} + \mathbf{C}_t\cdot\mathbf{L}\cdot\frac{\partial\mathbf{C}}{\partial\theta'}\cdot\mathbf{i}'p\theta'\Big) \quad \Big| \quad e_{\alpha'} = L_{\alpha\beta}C_{\alpha'}^{\alpha}C_{\beta'}^{\beta}\frac{di^{\beta'}}{dt} + L_{\alpha\beta}C_{\alpha'}^{\alpha}\frac{\partial C_{\beta'}^{\beta}}{\partial x^{\gamma'}}px^{\gamma'}i^{\beta'}$$

If $\mathbf{C}_t\cdot\mathbf{L}\cdot\mathbf{C} = \mathbf{L}'$ or $L_{\alpha\beta}C_{\alpha'}^{\alpha}C_{\beta'}^{\beta} = L_{\alpha'\beta'}$, then along the rotating reference frame the induced voltage becomes

$$\mathbf{e}' = \mathbf{L}'\cdot\frac{d\mathbf{i}'}{dt} + \mathbf{C}_t\cdot\mathbf{L}\cdot\frac{\partial\mathbf{C}}{\partial\theta'}\cdot\mathbf{i}'p\theta' \quad \Big| \quad e_{\alpha'} = L_{\alpha'\beta'}\frac{di^{\beta'}}{dt} + L_{\alpha\beta}C_{\alpha'}^{\alpha}\frac{\partial C_{\beta'}^{\beta}}{\partial x^{\gamma'}}px^{\gamma'}i^{\beta'} \qquad 26.5$$

(*b*) That is, along rotating reference frames *the previous induced voltage* $\mathbf{L}\cdot p\mathbf{i}$ *becomes partly an induced voltage and partly a generated voltage.* Hence the division of a voltage vector into induced and generated voltages is a relative concept that depends entirely on the reference frame. A certain voltage vector may be entirely induced or entirely generated voltage, or partly induced and partly generated, depending on the relative velocities of the reference frames, the fluxes, and the conductors. However, the *sum* of the induced and generated voltages is constant, no matter what the reference frame is.

It should be noted that the additional generated voltage $\mathbf{C}_t\cdot\mathbf{L}\cdot(\partial\mathbf{C}/\partial\theta')\cdot\mathbf{i}'p\theta'$ is different from the rotor-generated voltage $\mathbf{G}'\cdot\mathbf{i}'p\theta$. The former is due to the rotation of the *flux lines* produced by $\mathbf{i}'$ (the currents in the axes rotating with $p\theta'$); the latter, to the conductors rotating with $p\theta$ and cutting the flux lines produced by all currents in the machine.

The Equation of Voltage Along General Rotating Axes

The remaining part of $\mathbf{e} = \mathbf{Z}\cdot\mathbf{i}$, that is, $\mathbf{e} = \mathbf{R}\cdot\mathbf{i} + p\theta\,\mathbf{G}\cdot\mathbf{i}$, becomes after transformation

$$\mathbf{e}' = \mathbf{R}'\cdot\mathbf{i}' + p\theta\,\mathbf{G}'\cdot\mathbf{i}' \quad \Big| \quad e_{\alpha'} = R_{\alpha'\beta'}i^{\beta'} + p\theta\,G_{\alpha'\beta'}i^{\beta'} \qquad 26.6$$

Hence the equation of voltage along stationary axes

$$\mathbf{e} = \mathbf{R}\cdot\mathbf{i} + \mathbf{L}\cdot\frac{d\mathbf{i}}{dt} + p\theta\,\mathbf{G}\cdot\mathbf{i} \quad \Big| \quad e_\alpha = R_{\alpha\beta}i^\beta + L_{\alpha\beta}\frac{di^\beta}{dt} + p\theta\,G_{\alpha\beta}i^\beta \qquad 26.7$$

becomes after transformation with **C** into rotating axes

$$\left.\begin{array}{l}\boxed{\mathbf{e}' = \mathbf{R}'\cdot\mathbf{i}' + \mathbf{L}'\cdot\frac{d\mathbf{i}'}{dt} + p\theta\,\mathbf{G}'\cdot\mathbf{i}' + p\theta'\,\mathbf{V}'\cdot\mathbf{i}'} \\ \boxed{e_{\alpha'} = R_{\alpha'\beta'}i^{\beta'} + L_{\alpha'\beta'}\frac{di^{\beta'}}{dt} + p\theta\,G_{\alpha'\beta'}i^{\beta'} + p\theta'\,V_{\alpha'\beta'}i^{\beta'}}\end{array}\right\} \quad 26.8$$

where

$$\mathbf{V}' = \mathbf{C}_t^*\cdot\mathbf{L}\cdot\frac{\partial\mathbf{C}}{\partial\theta'}p\theta' \quad \Big| \quad V_{\alpha'\beta'} = L_{\alpha\beta}C^{\alpha}_{\alpha'}\frac{\partial C^{\beta}_{\beta'}}{\partial x^{\gamma'}}px^{\gamma'} \qquad 26.9$$

If the new voltage equation is written

$$\mathbf{e}' = \mathbf{Z}'\cdot\mathbf{i}' \quad \Big| \quad e_{\alpha'} = Z_{\alpha'\beta'}i^{\beta'} \qquad 26.10$$

then the law of transformation of **Z** follows as

$$\boxed{\mathbf{Z}' = \mathbf{C}_t^*\cdot\mathbf{Z}\cdot\mathbf{C} + \mathbf{C}_t^*\cdot\mathbf{L}\cdot\frac{\partial\mathbf{C}}{\partial\theta'}p\theta'} \quad \Big\| \quad \boxed{Z_{\alpha'\beta'} = Z_{\alpha\beta}C^{\alpha}_{\alpha'}C^{\beta}_{\alpha'} + L_{\alpha\beta}C^{\alpha}_{\alpha'}\frac{\partial C^{\beta}_{\beta'}}{\partial x^{\gamma'}}px^{\gamma'}} \quad 26.11$$

That is, both **Z** and **L** of the old reference frame have to be transformed.

In the equation $\mathbf{e}' = \mathbf{Z}'\cdot\mathbf{i}'$, any p refers only to $\mathbf{i}'$ and not to any $\cos\theta$ or $\sin\theta$ term occurring in $\mathbf{Z}'$.

The "Christoffel Symbol" $V_{\alpha\beta}$

It is possible to say that $\mathbf{V}'$ is a geometric object, called in tensor analysis the "Christoffel symbol" (strictly speaking, $V_{\alpha\beta}$ is only a special case of the true non-holonomic Christoffel symbol $[\alpha\beta,\gamma]$). Along the stationary **d** and **q** (quasi-holonomic) axes all the components of **V** happen to be zero but along rotating (non-holonomic) axes not all the components are zero. That is, the law of transformation of **V** is

$$\mathbf{V}' = \mathbf{C}_t^*\cdot\mathbf{V}\cdot\mathbf{C} + \mathbf{C}_t^*\cdot\mathbf{L}\cdot\frac{\partial\mathbf{C}}{\partial\theta'}p\theta' \quad \Big| \quad V_{\alpha'\beta'} = V_{\alpha\beta}C^{\alpha}_{\alpha'}C^{\beta}_{\beta'}\,L_{\alpha\beta}C^{\alpha}_{\alpha'}\frac{\partial C^{\beta}_{\beta'}}{\partial x^{\gamma'}}px^{\gamma'} \qquad 26.12$$

Since for the first primitive machine **V** is zero, therefore in going over to a rotating frame $\mathbf{C}_t^*\cdot\mathbf{V}\cdot\mathbf{C}$ is still zero, but $\mathbf{C}_t^*\cdot\mathbf{L}\cdot(\partial\mathbf{C}/\partial\theta')p\theta$ is not.

The "Rotation Tensor" γ_α^β

(*a*) It was shown in equation 16.24 that for synchronous and induction machines G can be derived from L with the aid of the "rotation tensor" γ.

$$\mathbf{G} = \gamma_t \cdot \mathbf{L} \quad \text{where} \quad \gamma_t = \begin{array}{c|cc} & d_r & q_r \\ \hline d_r & & 1 \\ q_r & -1 & \end{array} \tag{26.13}$$

(This assumption is true only if the flux-density wave is sinusoidal in space. In the general case [in commutator machines], the flux waves are non-sinusoidal, G is independent of L, and the rotation tensor γ has no existence.)

(*b*) Now the rotation tensor γ may be expressed in terms of C as follows:

$$\gamma = \frac{\partial \mathbf{C}}{\partial \theta} \cdot \mathbf{C}^{-1} \qquad \Big| \qquad \gamma_{\cdot\beta}^{\alpha} = \frac{\partial C_{\alpha'}^{\alpha}}{\partial \theta} C_{\beta}^{\alpha'} \tag{26.14}$$

$$\mathbf{C} = \begin{array}{c|cc} & a & b \\ \hline d_r & \cos\theta & -\sin\theta \\ q_r & \sin\theta & \cos\theta \end{array} \qquad \frac{\partial \mathbf{C}}{\partial \theta} = \begin{array}{c|cc} & a & b \\ \hline d_r & -\sin\theta & -\cos\theta \\ q_r & \cos\theta & -\sin\theta \end{array}$$

$$\mathbf{C}^{-1} = \begin{array}{c|cc} & d_r & q_r \\ \hline a & \cos\theta & \sin\theta \\ b & -\sin\theta & \cos\theta \end{array} \qquad \frac{\partial \mathbf{C}}{\partial \theta} \cdot \mathbf{C}^{-1} = \begin{array}{c|cc} & d_r & q_r \\ \hline d_r & & -1 \\ q_r & 1 & \end{array} = \gamma \tag{26.15}$$

Geometrically γ rotates a vector 90 degrees in space, as was shown in Fig. 16.9.

Consequently G may be expressed as

$$\boxed{\mathbf{G} = \gamma_t \cdot \mathbf{L} = \mathbf{C}_t^{-1} \cdot \frac{\partial \mathbf{C}}{\partial \theta} \cdot \mathbf{L}} \tag{26.16}$$

where C changes from stationary to rotating axes.

(*c*) It may be mentioned that the "rotation tensor" γ_α^β is a special case of the so-called "coefficients of rotation of Ricci" $\gamma_{\alpha\beta}^{\gamma}$ (just as $V_{\alpha\beta}$ is a special case of $\left\{ {\gamma \atop \alpha\beta} \right\}$). The reason for these simplified forms is that in the study of electrical machinery hitherto the rotor displace-

ment θ was not assumed as an extra variable, requiring an extra axis s, but as a parameter, since the speed $p\theta$ has been assumed to be constant. But as soon as the study of hunting and acceleration begins and an extra axis s has to be introduced (to express the equation of torque along it), both geometric objects of valence 2, γ_α^β and $V_{\alpha\beta}$, have to be replaced by their more general form as geometric objects of valence 3, $\gamma_{\alpha\beta}^\gamma$ and $\left\{{\gamma \atop \alpha\beta}\right\}$.

CHAPTER 27

TRANSFORMING THE TWO PRIMITIVE MACHINES INTO EACH OTHER

Equation of Voltage of Maxwell*

(*a*) In starting the analysis of synchronous or induction machines, the equations of either primitive machine may be used as a starting point, depending on which offers a speedier analysis. It will be shown now that the equations of the two types of machines can be derived from each other. For general commutator machines, however, the first primitive machine cannot be derived from the second, or vice versa.

(*b*) The equation of voltage along general rotating axes, equation 25.8, assumes a simple form if $p\theta' = p\theta$, that is, if the rotor axes rotate with the same speed as the rotor. It will now be proved that the two generated voltage terms may then be combined into one as

$$p\theta(\mathbf{G}' + \mathbf{V}')\cdot\mathbf{i}' = \frac{d\mathbf{L}'}{dt}\cdot\mathbf{i}' \qquad \Big| \qquad p\theta(G_{\alpha'\beta'} + V_{\alpha'\beta'})i^{\beta'} = \frac{dL_{\alpha'\beta'}}{dt}i^{\beta'} \qquad 27.1$$

or that

$$\mathbf{G}' + \mathbf{V}' = \partial\mathbf{L}'/\partial\theta \qquad \Big| \qquad G_{\alpha'\beta'} + V_{\alpha'\beta'} = \partial L_{\alpha'\beta'}/\partial\theta$$

(*c*) Since the rotation tensor $\boldsymbol{\gamma}$ can be expressed in terms of $\mathbf{C}$ by equation 26.14, for the *first* primitive machine

$$\mathbf{G} = \boldsymbol{\gamma}_t\cdot\mathbf{L} = \mathbf{C}_t^{*-1}\cdot\frac{\partial\mathbf{C}_t^*}{\partial\theta}\cdot\mathbf{L} \qquad \Big| \qquad G_{\alpha\beta} = \gamma^{\gamma}_{\cdot\alpha}L_{\gamma\beta} = \frac{\partial C^{\gamma}_{\alpha'}}{\partial\theta}C^{\alpha'}_{\alpha}L_{\gamma\beta} \qquad 27.2$$

For the second primitive machine $\mathbf{G}$ becomes

$$\mathbf{G}' = \mathbf{C}_t^*\cdot\mathbf{G}\cdot\mathbf{C} = \frac{\partial\mathbf{C}_t^*}{\partial\theta}\cdot\mathbf{L}\cdot\mathbf{C} \qquad \Big| \qquad G_{\alpha'\beta'} = G^{\alpha}_{\alpha\beta}C^{\alpha}_{\alpha'}C^{\beta}_{\beta'} = \frac{\partial C^{\gamma}_{\alpha'}}{\partial\theta}L_{\gamma\beta}C^{\beta}_{\beta'} \qquad 27.3$$

By equation 26.9

$$p\theta(\mathbf{G}' + \mathbf{V}') = p\theta\left[\frac{\partial\mathbf{C}_t^*}{\partial\theta}\cdot\mathbf{L}\cdot\mathbf{C} + \mathbf{C}_t^*\cdot\mathbf{L}\cdot\frac{\partial\mathbf{C}}{\partial\theta}\right]$$

$$= p\theta\frac{\partial(\mathbf{C}_t^*\cdot\mathbf{L}\cdot\mathbf{C})}{\partial\theta} = \frac{\partial\mathbf{L}'}{\partial\theta}\frac{\partial\theta}{\partial t} = \frac{d\mathbf{L}'}{dt} \qquad 27.4$$

* *A.T.E.M.*, p. 77.

if it is noted that $\partial \mathbf{L}/\partial\theta = 0$ ($\mathbf{L}$ of the first primitive machine has only constant components).

Hence for the second primitive machine the equation

$$\mathbf{e}' = \mathbf{R}'\cdot\mathbf{i}' + \mathbf{L}'\cdot p\mathbf{i}' + p\theta(\mathbf{G}' + \mathbf{V}')\cdot\mathbf{i}'$$

may be written in the form

$$\mathbf{e}' = \mathbf{R}'\cdot\mathbf{i}' + \mathbf{L}'\cdot p\mathbf{i}' + (p\mathbf{L}')\cdot\mathbf{i}' \qquad e_{\alpha'} = R_{\alpha'\beta'}i^{\beta'} + L_{\alpha'\beta'}pi^{\beta'} + (pL_{\alpha'\beta'})i^{\beta'}$$

$$\mathbf{e}' = \mathbf{R}'\cdot\mathbf{i}' + p(\mathbf{L}'\cdot\mathbf{i}') \qquad e_{\alpha'} = R_{\alpha'\beta'}i^{\beta'} + p(L_{\alpha'\beta'}i^{\beta'}) \qquad 27.5$$

or

$$\boxed{\mathbf{e}' = \mathbf{R}'\cdot\mathbf{i}' + p\varphi'} \qquad \boxed{e_{\alpha'} = R_{\alpha'\beta'}i^{\beta'} + p\varphi_{\alpha'}} \qquad 27.6$$

This is the equation with which Park starts to derive equation 26.7 for the synchronous machine along the direct and quadrature axes.

The Equation of Torque of Maxwell

The equation of torque

$$f' = \mathbf{i}'\cdot\mathbf{G}'\cdot\mathbf{i}' \qquad f' = G_{\alpha'\beta'}i^{\alpha'}i^{\beta'} \qquad 27.7$$

may be written by equation 27.3 as

$$f' = \mathbf{i}'\cdot\left(\frac{\partial \mathbf{C}_t}{\partial\theta}\cdot\mathbf{L}\cdot\mathbf{C}\right)\cdot\mathbf{i} \qquad f' = \frac{\partial C^{\gamma}_{\alpha'}}{\partial\theta}L_{\gamma\beta}C^{\beta}_{\beta'}i^{\alpha'}i^{\beta'} \qquad 27.8$$

Since in a quadratic form by equation 1.23

$$\mathbf{i}\cdot\mathbf{A}\cdot\mathbf{i} = \mathbf{i}\cdot\frac{\mathbf{A} + \mathbf{A}_t}{2}\cdot\mathbf{i} \qquad A_{\alpha\beta}i^{\alpha}i^{\beta} = \frac{A_{\alpha\beta} + A_{\beta\alpha}}{2}i^{\alpha}i^{\beta} \qquad 27.9$$

equation 27.8 may be written

$$f' = \frac{1}{2}\mathbf{i}'\cdot\left(\frac{\partial \mathbf{C}_t}{\partial\theta}\cdot\mathbf{L}\cdot\mathbf{C} + \mathbf{C}_t\cdot\mathbf{L}\cdot\frac{\partial \mathbf{C}}{\partial\theta}\right)\cdot\mathbf{i}' = \frac{1}{2}\mathbf{i}'\cdot\frac{\partial(\mathbf{C}_t\cdot\mathbf{L}\cdot\mathbf{C})}{\partial\theta}\cdot\mathbf{i}'$$

$$f' = \frac{1}{2}\mathbf{i}'\cdot\frac{\partial \mathbf{L}'}{\partial\theta}\cdot\mathbf{i}' \qquad f' = \frac{1}{2}\frac{\partial L_{\alpha'\beta'}}{\partial\theta}i^{\alpha'}i^{\beta'} \qquad 27.10$$

(Again it should be remembered that $\partial\mathbf{L}/\partial\theta = 0$ as the components of $\mathbf{L}$ are constant.) Since the instantaneous kinetic energy (magnetic energy) stored in the machine is

$$T' = \tfrac{1}{2}\,\mathbf{i}'\cdot\mathbf{L}'\cdot\mathbf{i}' \qquad T' = \tfrac{1}{2}\,L_{\alpha'\beta'}i^{\alpha'}i^{\beta'} \qquad 27.11$$

therefore

$$\boxed{f' = \frac{\partial T'}{\partial\theta}} \qquad \boxed{f' = \frac{\partial T'}{\partial\theta}} \qquad 27.12$$

Equation of Voltage of the First Primitive Machine *

(*a*) The reverse of the previous derivation, to be shown now, is identical with that given by Park.

The equation of voltage of Maxwell for the primitive machine with rotating axes is

$$\mathbf{e} = \mathbf{R}\cdot\mathbf{i} + p(\mathbf{L}\cdot\mathbf{i}) \qquad \bigg| \qquad e_m = R_{mn}i^n + p(L_{mn}i^n) \qquad 27.13$$

where $\mathbf{L}$ is given in equation 24.6.

Now let the rotating reference frame $\mathbf{a}$ and $\mathbf{b}$ be replaced by stationary reference axes $\mathbf{d}$ and $\mathbf{q}$ by the transformation $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$.

$$\begin{aligned} i^{d_s} &= i^{d_s} \\ i^a &= i^{d_r}\cos\theta + i^{q_r}\sin\theta \\ i^b &= -i^{d_r}\sin\theta\, i^{q_r}\cos\theta \\ i^{q_s} &= i^{q_s} \end{aligned}$$

$$\mathbf{C} = \begin{array}{c|cccc|} & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\ \hline \mathbf{d}_s & 1 & & & \\ \mathbf{a} & & \cos\theta & \sin\theta & \\ \mathbf{b} & & -\sin\theta & \cos\theta & \\ \mathbf{q}_s & & & & 1 \\ \hline \end{array} \qquad 27.14$$

Note that this $\mathbf{C}$ is the inverse of what formerly in equation 24.4 was called $\mathbf{C}$ (Fig. 27.1).

(Or using the convention of central-station engineers, Fig. 27.2,

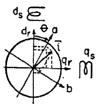


FIG. 27.1. Relation between stationary and rotating axes.

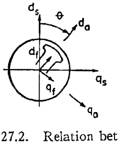


FIG. 27.2. Relation between axes in a synchronous machine.

let the stationary axes $\mathbf{d}_s$ and $\mathbf{q}_s$ on the armature be replaced by axes $\mathbf{d}_a$ and $\mathbf{q}_a$ rotating with the same speed as the field pole.)

$$\mathbf{C} = \begin{array}{c|cccc|} & \mathbf{d}_f & \mathbf{d}_a & \mathbf{q}_a & \mathbf{q}_f \\ \hline \mathbf{d}_f & 1 & & & \\ \mathbf{d}_s & & \cos\theta & -\sin\theta & \\ \mathbf{q}_s & & \sin\theta & \cos\theta & \\ \mathbf{q}_f & & & & 1 \\ \hline \end{array} \qquad 27.15$$

* *G.E.R.*, May, 1938, p. 244.

(*b*) Substituting $\mathbf{C}\cdot\mathbf{i}'$ for $\mathbf{i}$,

$$\mathbf{e} = \mathbf{R}\cdot\mathbf{C}\cdot\mathbf{i}' + p(\mathbf{L}\cdot\mathbf{C}\cdot\mathbf{i}')$$

$$= \mathbf{R}\cdot\mathbf{C}\cdot\mathbf{i}' + p(\mathbf{L}\cdot\mathbf{C})\cdot\mathbf{i}' + \mathbf{L}\cdot\mathbf{C}\cdot p\mathbf{i}'$$

Let both sides of the equations be multiplied by $\mathbf{C}_t$,

$$\mathbf{C}_t\cdot\mathbf{e} = \mathbf{C}_t\cdot\mathbf{R}\cdot\mathbf{C}\cdot\mathbf{i}' + \mathbf{C}_t\cdot p(\mathbf{L}\cdot\mathbf{C})\cdot\mathbf{i}' + \mathbf{C}_t\cdot\mathbf{L}\cdot\mathbf{C}\cdot p\mathbf{i}'$$

But

$$\mathbf{C}_t\cdot\mathbf{e} = \mathbf{e}'$$

$$\mathbf{C}_t\cdot\mathbf{R}\cdot\mathbf{C} = \mathbf{R}' \qquad 27.16$$

$$\mathbf{C}_t\cdot\mathbf{L}\cdot\mathbf{C} = \mathbf{L}'$$

where the primed quantities represent the tensors of the primitive machine with stationary axes. Hence

$$\mathbf{e}' = \mathbf{R}'\cdot\mathbf{i}' + \mathbf{L}'\cdot p\mathbf{i}' + \mathbf{C}_t\cdot p(\mathbf{L}\cdot\mathbf{C})\cdot\mathbf{i}' \qquad 27.17$$

(*c*) The expression $p(\mathbf{L}\cdot\mathbf{C})$ can be brought to a more recognizable form by replacing $\mathbf{L}$ by

$$\mathbf{L} = \mathbf{C}_t^{-1}\cdot\mathbf{L}'\cdot\mathbf{C}^{-1}$$

where $\mathbf{L}'$ is given in equation 24.5. Then

$$p(\mathbf{L}\cdot\mathbf{C}) = p(\mathbf{C}_t^{-1}\cdot\mathbf{L}') = (p\mathbf{C}_t^{-1})\cdot\mathbf{L}'$$

since $p\mathbf{L}'$ is zero (all components of $\mathbf{L}'$ being constant).

Since $\mathbf{C}$ and $\mathbf{C}^{-1}$ are functions of θ,

$$p(\mathbf{L}\cdot\mathbf{C}) = \left(\frac{d}{dt}\,\mathbf{C}_t^{-1}\right)\cdot\mathbf{L}' = \frac{\partial\mathbf{C}_t^{-1}}{\partial\theta}\cdot\mathbf{L}'\frac{d\theta}{dt}$$

Substituting into equation 27.17

$$\mathbf{e}' = \mathbf{R}'\cdot\mathbf{i}' + \mathbf{L}'\cdot p\mathbf{i}' + \mathbf{C}_t\cdot\frac{\partial\mathbf{C}_t^{-1}}{\partial\theta}\cdot\mathbf{L}'\,p\theta\cdot\mathbf{i}'$$

But by equations 26.14 and 26.13

$$\mathbf{C}_t\cdot\frac{\partial\mathbf{C}_t^{-1}}{\partial\theta} = \boldsymbol{\Upsilon}_t \qquad 27.18$$

$$\boxed{\mathbf{G}' = \boldsymbol{\Upsilon}_t\cdot\mathbf{L}' = \mathbf{C}_t\cdot\frac{\partial\mathbf{C}_t^{-1}}{\partial\theta}\cdot\mathbf{L}'} \qquad 27.19$$

(where $\mathbf{C}$ changes from rotating to stationary axes). The equation of voltage of the primitive machine becomes

$$\mathbf{e}' = \mathbf{R}'\cdot\mathbf{i}' + \mathbf{L}'\cdot p\mathbf{i}' + p\theta\,\mathbf{G}'\cdot\mathbf{i}' \qquad 27.20$$

This is the same as equation 26.7.

Equation of Torque of the First Primitive Machine

Let the equation of torque of Maxwell for the primitive machine with rotating axes be

$$f = \frac{\partial T}{\partial\theta} = \frac{1}{2}\mathbf{i}\cdot\frac{\partial\mathbf{L}}{\partial\theta}\cdot\mathbf{i} \qquad 27.21$$

Let $\mathbf{i}$ be replaced by $\mathbf{C}\cdot\mathbf{i}' = \mathbf{i}'\cdot\mathbf{C}_t$

$$f = \frac{1}{2}\mathbf{i}'\cdot\mathbf{C}_t\cdot\frac{\partial\mathbf{L}}{\partial\theta}\cdot\mathbf{C}\cdot\mathbf{i}'$$

Again replacing $\mathbf{L}$ by $\mathbf{C}_t^{-1}\cdot\mathbf{L}'\cdot\mathbf{C}^{-1}$,

$$f = \frac{1}{2}\mathbf{i}'\cdot\mathbf{C}_t\cdot\frac{\partial(\mathbf{C}_t^{-1}\cdot\mathbf{L}'\cdot\mathbf{C}^{-1})}{\partial\theta}\cdot\mathbf{C}\cdot\mathbf{i}'$$

Since $\mathbf{L}'$ is constant, $\partial\mathbf{L}'/\partial\theta$ is zero. Hence

$$f = \frac{1}{2}\mathbf{i}'\cdot\mathbf{C}_t\cdot\frac{\partial\mathbf{C}_t^{-1}}{\partial\theta}\cdot\mathbf{L}'\cdot\mathbf{i}' + \frac{1}{2}\mathbf{i}'\cdot\mathbf{L}'\cdot\frac{\partial\mathbf{C}^{-1}}{\partial\theta}\cdot\mathbf{C}\cdot\mathbf{i}'$$

However, the second term is equal to the first since $\mathbf{i}\cdot\mathbf{A}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{A}_t\cdot\mathbf{i}$ and $\mathbf{L}_t = \mathbf{L}$. Hence

$$f = \mathbf{i}'\cdot\mathbf{C}_t\cdot\frac{\partial\mathbf{C}_t^{-1}}{\partial\theta}\cdot\mathbf{L}'\cdot\mathbf{i}'$$

Since by equation 27.19

$$\mathbf{C}_t\cdot\frac{\partial\mathbf{C}_t^{-1}}{\partial\theta}\cdot\mathbf{L}' = \mathbf{G}' \qquad 27.22$$

therefore the torque of the primitive machine is

$$f = \mathbf{i}'\cdot\mathbf{G}'\cdot\mathbf{i}' \qquad 27.23$$

CHAPTER 28

SMALL OSCILLATIONS*

The Equations of Voltages and Torques

(*a*) During small oscillations (hunting) the speed of the rotor $p\theta$ is no more constant and the moment of inertia M of the rotor enters into the equation of torque. The equations of *impressed* voltage and *impressed* torque of the first primitive machine are

$$\mathbf{e} = \mathbf{R}\cdot\mathbf{i} + \mathbf{L}\cdot p\mathbf{i} + p\theta\,\mathbf{G}\cdot\mathbf{i} \qquad e_m = R_{mn}i^n + L_{mn}pi^n + p\theta\, G_{mn}i^n$$

$$T = Mp^2\theta - \mathbf{i}\cdot\mathbf{G}\cdot\mathbf{i} \qquad T = Mp^2\theta - G_{mn}i^m i^n \tag{28.1}$$

These two equations describe the performance of the primitive machine (hence all machines with relatively stationary axes) during acceleration. In terms of φ and $\mathbf{B}$ they are

$$\mathbf{e} = \mathbf{R}\cdot\mathbf{i} + p\varphi + \mathbf{B}\,p\theta \qquad e_m = R_{mn}i^n + p\varphi_m + B_m p\theta$$

$$T = Mp^2\theta - \mathbf{i}\cdot\mathbf{B} \qquad T = Mp^2\theta - i^nB_n \tag{28.2}$$

(*b*) When the machine's equilibrium is suddenly disturbed, $\mathbf{i}$ becomes $\mathbf{i}_0 + \Delta\mathbf{i}$, where $\mathbf{i}_0$ represents the steady-state current existing before the disturbance, and $\Delta\mathbf{i}$ the superimposed change. Let

$$\mathbf{i} = \mathbf{i}_0 + \Delta\mathbf{i} \qquad \mathbf{e} = \mathbf{e}_0 + \Delta\mathbf{e}$$

$$\theta = \theta_0 + \Delta\theta \qquad T = T_0 + \Delta T \tag{28.3}$$

The tensors $\mathbf{R}$, $\mathbf{L}$, and $\mathbf{G}$ have constant components; hence, no change occurs in them during hunting.

Substituting and canceling second-order changes,

$$\mathbf{e}_0 + \Delta\mathbf{e} = (\mathbf{R} + \mathbf{L}p + p\theta\,\mathbf{G})\cdot(\mathbf{i}_0 + \Delta\mathbf{i}) + p\Delta\theta\,\mathbf{G}\cdot\mathbf{i}_0$$

$$T_0 + \Delta T = Mp^2(\theta_0 + \Delta\theta) - (\mathbf{i}_0 + \Delta\mathbf{i})\cdot\mathbf{G}\cdot(\mathbf{i}_0 + \Delta\mathbf{i}) \tag{28.4}$$

Subtracting the original equations (and assuming $p\Delta\theta = \Delta p\theta$), the *equations of hunting* of the primitive machine are

$$\Delta\mathbf{e} = (\mathbf{R} + \mathbf{L}p + p\theta\,\mathbf{G})\cdot\Delta\mathbf{i} + \mathbf{G}\cdot\mathbf{i}_o\,\Delta p\theta$$

$$\Delta T = Mp^2\Delta\theta - \mathbf{i}_0\cdot(\mathbf{G} + \mathbf{G}_t)\cdot\Delta\mathbf{i} \tag{28.5}$$

* *A.T.E.M.*, p. 114.

In terms of φ and B the above equations are

$$\Delta e = R\cdot\Delta i + p\Delta\varphi + B_0\Delta p\theta$$
$$\Delta T = Mp^2\Delta\theta - (B_0\cdot\Delta i + \Delta B\cdot i_0) \qquad 28.6$$

The Motional Impedance Tensor

(*a*) The two equations may be combined into one if, in addition to the four electrical axes d_s, d_r, q_r, and q_s, a fifth axis s is introduced representing the direction of the instantaneous angular displacement θ of the rotor (geometrically s lies along the rotor axis). All torques are represented along this geometrical axis s. In the presence of a fifth variable θ, let "compound tensors" be introduced. In particular let:

1. **i** and $p\theta$ be represented as the components of a new "generalized velocity (or current) vector" $\dot{x}$

$$\dot{x} = \begin{array}{|c|c|}\hline i & p\theta \\ \hline\end{array} = \begin{array}{ccccc} d_s & d_r & q_r & q_s & s \\ \begin{array}{|c|}\hline i^{d_s} \\ \hline\end{array} & \begin{array}{|c|}\hline i^{d_r} \\ \hline\end{array} & \begin{array}{|c|}\hline i^{q_r} \\ \hline\end{array} & \begin{array}{|c|}\hline i^{q_s} \\ \hline\end{array} & \begin{array}{|c|}\hline p\theta \\ \hline\end{array} \end{array} \qquad 28.7$$

2. e and T be represented as the components of a new "generalized force (or voltage) vector" p

$$p = \begin{array}{|c|c|}\hline e & T \\ \hline\end{array} = \begin{array}{ccccc} d_s & d_r & q_r & q_s & s \\ \begin{array}{|c|}\hline e_{d_s} \\ \hline\end{array} & \begin{array}{|c|}\hline e_{d_r} \\ \hline\end{array} & \begin{array}{|c|}\hline e_{q_r} \\ \hline\end{array} & \begin{array}{|c|}\hline e_{q_s} \\ \hline\end{array} & \begin{array}{|c|}\hline T \\ \hline\end{array} \end{array} \qquad 28.8$$

(*b*) In terms of these generalized vectors, the two equations of hunting 28.5 may be represented as subdivisions (in the manner of equation 2.1) of *one* equation $\Delta p = \mathbf{Z}\cdot\Delta\dot{x}$

$$\begin{array}{ccccc} & & \begin{array}{|c|c|}\hline \Delta i & \Delta p\theta \\ \hline\end{array} & & \begin{array}{|c|c|}\hline \Delta i & \Delta p\theta \\ \hline\end{array} \\ & & \cdot & & \cdot \\ \begin{array}{|c|}\hline \Delta e \\ \hline \Delta T \\ \hline\end{array} & = & \begin{array}{|c|c|}\hline R + Lp + p\theta G & G\cdot i_0 \\ \hline -i_0\cdot(G + G_t) & Mp \\ \hline\end{array} & = & \begin{array}{|c|c|}\hline Z & B_0 \\ \hline -i_0\cdot G_t - B_0 & Mp \\ \hline\end{array} \end{array} \qquad 28.9$$

where $\mathbf{Z}$ will be called the "motional impedance tensor."

The motional impedance tensor $\mathbf{Z}$ *for any machine consists of its transient impedance tensor* Z *augmented by an additional row and column* s *corresponding to the additional (geometrical) degree of freedom.* The additional row and column contain the steady-state currents and fluxes upon which the hunting is superimposed.

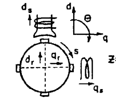
Table VII shows $\mathbf{Z}$ and $\mathbf{Z}_g$ of the primitive machine for various sign conventions.

It is important to note that in the additional row the flux densities

TABLE VII

MOTIONAL-IMPEDANCE TENSOR $\mathbf{Z}$ OF THE PRIMITIVE MACHINE

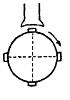
1



$\mathbf{Z} =$

	d_s	d_r	q_r	q_s	s
d_s	$r_{ds}+L_{ds}p$	M_dp	0	0	0
d_r	M_dp	$r_r+L_{dr}p$	$L'_{qr}p\theta$	$M'_qp\theta$	B_{dr}
q_r	$-M'_dp\theta$	$-L'_{dr}p\theta$	$r_r+L_{qr}p$	M_qp	B_{qr}
q_s	0	0	M_qp	$r_{qs}+L_{qs}p$	0
s	$i^{qr}M'_d$	$-B_{dr}+i^{qr}L'_{dr}$	$-B_{qr}-i^{dr}L'_{qr}$	$-i^{dr}M'_q$	Lp

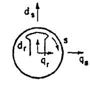
2



$\mathbf{Z} =$

	d_s	q_s	d_r	q_r	s
d_s	$r_{ds}+L_{ds}p$	0	M_dp	0	0
q_s	0	$r_{qs}+L_{qs}p$	0	M_qp	0
d_r	M_dp	$M'_qp\theta$	$r_r+L_{dr}p$	$L'_{qr}p\theta$	B_{dr}
q_r	$-M'_dp\theta$	M_qp	$-L'_{dr}p\theta$	$r_r+L_{qr}p$	B_{qr}
s	$i^{qr}M'_d$	$-i^{dr}M'_q$	$-B_{dr}+i^{qr}L'_{dr}$	$-B_{qr}-i^{dr}L'_{qr}$	Lp

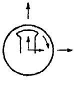
3



$\mathbf{Z} =$

	d_r	d_s	q_s	q_r	s
d_r	$r_{dr}+L_{dr}p$	M_dp	0	0	0
d_s	M_dp	$r_s+L_{ds}p$	$-L'_{qs}p\theta$	$-M'_qp\theta$	$-B_{ds}$
q_s	$M'_dp\theta$	$L'_{ds}p\theta$	$r_s+L_{qs}p$	M_qp	$-B_{qs}$
q_r	0	0	M_qp	$r_{qr}+L_{qr}p$	0
s	$-i^{qs}M'_d$	$B_{ds}-i^{qs}L'_{ds}$	$B_{qs}+i^{ds}L'_{qs}$	$i^{ds}M'_q$	Lp

4



$\mathbf{Z}_g =$

	d_r	d_s	q_s	q_r	s
d_r	$-r_{dr}-L_{dr}p$	$-M_dp$	0	0	0
d_s	$-M_dp$	$-r_s-L_{ds}p$	$L'_{qs}p\theta$	$M'_qp\theta$	B_{ds}
q_s	$-M'_dp\theta$	$-L'_{ds}p\theta$	$-r_s-L_{qs}p$	$-M_qp$	B_{qs}
q_r	0	0	$-M_qp$	$-r_{qr}-L_{qr}p$	0
s	$i^{qs}M'_d$	$-B_{ds}+i^{qs}L'_{ds}$	$-B_{qs}-i^{ds}L'_{qs}$	$-i^{ds}M'_q$	$-Lp$

5

$B_{dr} = i^{qr}L'_{qr} + i^{qs}M'_q$ $\qquad$ $B_{ds} = i^{qs}L'_{qs} + i^{qr}M'_q$

$B_{qr} = -i^{dr}L'_{dr} - i^{ds}M'_d$ $\qquad$ $B_{qs} = -i^{ds}L'_{ds} - i^{dr}M'_d$

B_d and B_q occur with *signs opposite* to those in the additional column, no matter what sign convention is used (as long as the coefficients of all p terms—the components of $a_{\alpha\beta}$—have the same sign). That is, *$\mathbf{Z}$ is always skew symmetrical with respect to* $\mathbf{B}$ *in any reference frame.* (See also equation 31.4.) This relation serves as a check on the correctness of the equations.

(*c*) It should also be noted that when the direction of rotation changes, then:

1. $p\theta$ assumes negative values.
2. The row and column of $\mathbf{s}$ are also multipled by -1.

The Establishment of $\mathbf{Z}'$

(*a*) Since the $\mathbf{Z}$ of the primitive machine and of every other machine has an extra axis $\mathbf{s}$, similarly the $\mathbf{C}$ of every machine has a geometrical axis $\mathbf{s}$ in addition to its electrical axes.

When all components of $\mathbf{C}$ are constants, then $\mathbf{Z}' = \mathbf{C}_t^* \cdot \mathbf{Z} \cdot \mathbf{C}$. Since $\mathbf{Z}$ contains the steady-state currents $\mathbf{i}_0$ of the primitive machine, *after transforming* $\mathbf{Z}$ *by* $\mathbf{C}_t^* \cdot \mathbf{Z} \cdot \mathbf{C}$, *it is still necessary to transform the steady-state currents individually with the aid of the set of equations* $\mathbf{i}_0 = \mathbf{C} \cdot \mathbf{i}_0'$. Thereby not only $\mathbf{Z}'$ is expressed along the new axes but also its components.

(*b*) Once the transient $\mathbf{Z}'$ of a machine has been established, it may be subjected to various types of manipulation depending on the problem at hand. In particular, may be investigated:

1. The stability of the system under a sudden impact of voltage or torque.
2. The values of the hunting-frequency currents and displacements under impressed impulses.
3. The damping and synchronizing torques.
4. The natural frequencies of vibration of the system.

In all such investigations the first step is to establish the transient motional impedance tensor $\mathbf{Z}'$ of the system.

Transient Stability

The stability of the system under a sudden impact is investigated by equating the determinant of $\mathbf{Z}'$ to zero and applying Routh's or other criteria.

Even when the components of $\mathbf{C}$ are constants, two cases will have to be distinguished.

1. The steady-state currents $\mathbf{i}_0$ are constants.
2. The steady-state currents are complex numbers (sinusoidal in time).

In the first case the coefficients of all p are *real* numbers; in the second, they are complex. In the first case Routh's criterion, shown in equations 19.7–19.10, in the second case Schur's criterion (given in advanced mathematical textbooks), have to be used.

Hunting-Frequency Currents and Velocities

When the impressed changes $\Delta p'$ are sinusoidal (say when the machine drives a pump with sinusoidal load variation) having a frequency h, then the additional currents and velocities may be determined. Again two cases have to be distinguished.

(*a*) When the steady-state currents $\mathbf{i}_0$ are constant, all $\Delta \dot{x}$ are of hunting frequency h, hence:

1. All p are replaced by $jh\omega$.
2. All $p\theta$ become $v\omega$.
3. $\Delta p\theta$ becomes $\Delta v\omega$, or rather the last column s is multiplied by ω, changing there all L to X and leaving Δv as the variable.
4. To express torques in synchronous watts, the last row of s is also multiplied by ω.

Then $\Delta \dot{\mathbf{x}}'$ is found by $\mathbf{Z}'^{-1} \cdot \Delta \mathbf{p}$, that is, by calculating the inverse of the steady state $\mathbf{Z}$.

(*b*) When the steady-state currents $\mathbf{i}_0$ are not constant but are, say, of fundamental frequency ω, then the superimposed currents have two different frequencies $(1 - h)\omega$ and $(1 + h)\omega$. The solution of $\mathbf{Z}$ for such cases has been undertaken in another publication.*

Damping and Synchronizing Torques

To determine the stability or instability of a machine, the determinant of $\mathbf{Z}$ is equated to zero and Routh's criterion (equation 19.7) is applied. Another method of analysis is based upon the assumption that only one dominating oscillation frequency $h\omega$ exists (whose approximate value, however, has to be assumed).

Leaving out Mp^2 from $\mathbf{Z}$ and subdividing $\mathbf{Z}$ along the electrical and mechanical axes into four components, the applied electrical torque is a complex number (replacing all p by $jh\omega$)

$$\Delta T_e = (\mathbf{Z}_4 - \mathbf{Z}_3 \cdot \mathbf{Z}_1^{-1} \cdot \mathbf{Z}_2)\Delta\theta = (T_s + jhT_D)\Delta\theta \qquad 28.10$$

T_D is called the damping torque coefficient, and T_s the synchronizing torque coefficient. *When T_D is negative the system hunts.*

Natural Frequency of Oscillation

Once T_D and T_s are known, then Mp^2 can be resubstituted, giving (for a single machine)

$$\Delta T = (Mp^2 + T_D p + T_s)\Delta\theta \qquad 28.11$$

* *A.T.E.M.*, p. 119.

The equation $Mp^2 + T_D p + T_s = 0$ gives for the natural frequency of oscillation (as a fraction of ω)

$$h = \sqrt{\frac{T_s}{M} - \left(\frac{T_D}{2M}\right)^2} \tag{28.12}$$

where in per unit $M = 4\pi f H$ and

$$H = \frac{0.231\ (WR^2) \times (\text{syn. r.p.m.})^2}{\text{Base kv-a.} \times 10^6} \tag{28.13}$$

When T_D is small

$$h = \sqrt{\frac{T_s}{M}} \tag{28.14}$$

If in calculating T_s the assumed h differs greatly from this correct h, T_s should be recalculated with the corrected h.

Compound D-C. Machine

The connection diagram and $\mathbf{C}$ of a compound d-c. machine are (Fig. 28.1)

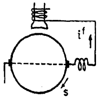


FIG. 28.1 Compound d-c. machine.

$$\mathbf{C} = \begin{array}{c|c|c|} & \mathrm{f} & \mathrm{s} \\ \hline \mathrm{d}_s & n_d & \\ \hline \mathrm{q}_r & 1 & \\ \hline \mathrm{q}_s & n_q & \\ \hline \mathrm{s} & & 1 \\ \hline \end{array} \tag{28.15}$$

$\mathbf{Z}$ of the primitive machine is, from Table VII–1,

$$\mathbf{Z} = \begin{array}{c|c|c|c|c|} & \mathrm{d}_s & \mathrm{q}_r & \mathrm{q}_s & \mathrm{s} \\ \hline \mathrm{d}_s & r_{ds} + L_{ds}p & & & \\ \hline \mathrm{q}_r & -M'_d p\theta & r_r + L_{qr}p & M_q p & -i^{ds}M'_d \\ \hline \mathrm{q}_s & & M_q p & r_{qs} + L_{qs}p & \\ \hline \mathrm{s} & i^{qr}M'_d & i^{ds}M'_d & & Mp \\ \hline \end{array}$$

The transient $\mathbf{Z}'$ is found by $\mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C}$. Replacing in the border row and column i^{ds} by $n_d i^f$ and i^{qr} by i^f, as indicated by equation 28.15,

$$\mathbf{Z}' = \begin{array}{c|c|c|} & \mathrm{f} & \mathrm{s} \\ \hline \mathrm{f} & \begin{array}{l} n_d^2(r_{ds} + L_{ds}p) + (r_r + L_{qr}p) + \\ n_q^2(r_{qs} + L_{qs}p) - n_d M_d' p\theta + 2n_q M_q p \end{array} & -i^f n_d M_d' \\ \hline \mathrm{s} & 2i^f n_d M_d' & Mp \\ \hline \end{array}$$

The steady-state $\mathbf{Z}'$ is found by putting $p = jh\omega$, $p\theta = v\omega = \omega$ and multiplying the bordering row and column s by ω

$$\mathbf{Z}' = \begin{array}{c|c|c|} & \mathrm{f} & \mathrm{s} \\ \hline \mathrm{f} & \begin{array}{l} n_d^2(r_{ds} + jhX_{ds}) + (r_r + jhX_{qr}) + \\ n_q^2(r_{qs}\, jhX_{qs}) - n_d X_{md}' + 2n_q jX_{mq} \end{array} & -i^f n_d X_{md}' \\ \hline \mathrm{s} & 2i^f n_d X_{md}' & jh\omega M \\ \hline \end{array} \qquad 28.16$$

The equations of hunting are $\Delta \mathbf{p}' = \mathbf{Z}' \cdot \Delta \dot{\mathbf{x}}$.

Polyphase Induction Motor

It has been shown for the double-fed induction motor that a reference frame rotating with the fluxes allows a simpler steady-state and transient analysis of polyphase machines. The use of such a reference frame during hunting makes the steady-state currents and fluxes in $\mathbf{Z}$ constant.

The same $\mathbf{C}$ is used in transforming $\mathbf{Z}$ as used for $\mathbf{Z}$, namely, equation 23.9, except that $\mathbf{C}$ now has an additional axis s

$$\mathbf{C} = \begin{array}{c|c|c|c|c|c|} & \mathrm{a}_s & \mathrm{a}_r & \mathrm{b}_r & \mathrm{b}_s & \mathrm{s} \\ \hline \mathrm{d}_s & \cos\theta_1 & & & -\sin\theta_1 & \\ \hline \mathrm{d}_r & & \cos\theta_1 & -\sin\theta_1 & & \\ \hline \mathrm{q}_r & & \sin\theta_1 & \cos\theta_1 & & \\ \hline \mathrm{q}_s & \sin\theta_1 & & & \cos\theta_1 & \\ \hline \mathrm{s} & & & & & 1 \\ \hline \end{array} \qquad 28.17$$

Since $\mathbf{C}$ is a function of time, the law of transformation of $\mathbf{Z}$ is found either by $\mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C}$, where the p in $\mathbf{Z}$ refers to $\mathbf{C}$ (but not to $\mathbf{C}_t$), or by equation 23.3

$$\mathbf{Z}' = \mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C} + \mathbf{C}_t \cdot \mathbf{L} \cdot \frac{\partial \mathbf{C}}{\partial \theta_1} p\theta_1$$

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \mathbf{a}_s & \mathbf{a}_r & \mathbf{b}_r & \mathbf{b}_s & \mathbf{s} \end{array} \\
\mathbf{Z}' = \begin{array}{c} \mathbf{a}_s \\ \mathbf{a}_r \\ \mathbf{b}_r \\ \mathbf{b}_s \\ \mathbf{s} \end{array} &
\left|\begin{array}{c|c|c|c|c}
r_s + L_s p & Mp & -Mp\theta_1 & -L_s p\theta_1 & \\
\hline
Mp & r_r + L_r p & -L_r p\theta_s & -Mp\theta_s & i^{bs}M + i^{br}L_r \\
\hline
Mp\theta_s & L_r p\theta_s & r_r + L_r p & Mp & -i^{as}M - i^{ar}L_r \\
\hline
L_s p\theta_1 & Mp\theta_1 & Mp & r_s + L_s p & \\
\hline
i^{br}M & -i^{bs}M & i^{as}M & -i^{ar}M & Lp
\end{array}\right|
\end{array}
\qquad 28.18
$$

where $p\theta_s = p\theta_1 - p\theta_2$. All steady-state currents are constant.

When no voltage is impressed on the rotor, as in a standard induc-

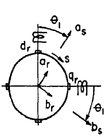


FIG. 28.2. Axes rotating with stator flux.

tion motor, this $\mathbf{Z}$ is used unchanged. However, in a double-fed motor $\mathbf{e}$, equation 23.7 is a function of δ and an additional term has to be added to $\mathbf{Z}$, as will be shown in equation 29.20.

EXERCISES

1. Eliminate the stator axes and stator currents in $\mathbf{Z}$ of equation 28.18.
2. Find the steady-state form of equation 28.18.
3. Find the transient and steady-state $\mathbf{Z}$ of the following machines:
   (*a*) The amplidyne of Fig. 18.2*b*.
   (*b*) The Scherbius advancer of Fig. 22.6.
   (*c*) The shunt polyphase commutator motor of Fig. 22.10.
   (*d*) The double squirrel-cage induction motor of Fig. 22.8.

CHAPTER 29

THE HUNTING OF MACHINES WITH SLIP RINGS

Calculation of $\Delta\mathbf{p}'$

(*a*) The steady-state voltage impressed on a machine is $\mathbf{e}' = \mathbf{C}_t \cdot \mathbf{e}$. If its components are constant, then $\mathbf{e}'$ does not contribute to $\Delta\mathbf{e}'$. But if $\mathbf{e}'$ is a function of δ or θ (as it is in all machines having slip rings), then during hunting its contribution to $\Delta\mathbf{e}'$ is

$$\Delta\mathbf{e}' = \frac{\partial \mathbf{e}'}{\partial \boldsymbol{\theta}} \cdot \Delta\boldsymbol{\theta} \qquad \Big| \qquad \Delta e^{m'} = \frac{\partial e^{m'}}{\partial x^{n'}} \Delta x^{n'} \qquad 29.1$$

In general the value $\Delta\mathbf{p}'$ in the equation $\Delta\mathbf{p}' = \mathbf{Z}' \cdot \Delta\mathbf{x}'$ is

$$\Delta\mathbf{p}' = \frac{\partial \mathbf{p}'}{\partial \boldsymbol{\theta}} \Delta\boldsymbol{\theta} + \mathbf{P}' \qquad \Big| \qquad \Delta p_{m'} = \frac{\partial p_{m'}}{\partial x^{n'}} \Delta x^{n'} \qquad 29.2$$

where $\mathbf{p}' = \mathbf{C}_t \cdot \mathbf{p}$, and where

1. $(\partial\mathbf{p}'/\partial\boldsymbol{\theta}) \cdot \Delta\boldsymbol{\theta}$ is due to the presence of applied variable steady-state voltages and torques.
2. $\mathbf{P}'$ is any additional sudden or hunting-frequency change of voltage or torque applied.

(*b*) In order to represent the equations of hunting also in this case as $\Delta\mathbf{p}'' = \mathbf{Z}'' \cdot \Delta\dot{\mathbf{x}}''$, the $\Delta\theta$ term of $\Delta e_g'$ is carried over to the right-hand side of the equation. Since on the right-hand side the column of $\Delta p\theta$ already occurs, *in such cases* $\Delta\theta$ *is assumed as the variable in place of* $\Delta p\theta$ *and the corresponding column of* $\mathbf{Z}'$ *(after transformation) is multiplied by* p. Then the two columns of $\Delta\theta$ can be added to form a new column of $\mathbf{Z}''$. That is, now the law of transformation of $\mathbf{Z}$ is

$$\boxed{\mathbf{Z}' = \mathbf{C}_t^* \cdot \mathbf{Z} \cdot \mathbf{C} - \frac{\partial \mathbf{p}'}{\partial \theta}} \qquad \Big| \qquad \boxed{Z_{\alpha'\beta'} = Z_{\alpha\beta} C_{\alpha'}^{\alpha} C_{\beta'}^{\beta} - \frac{\partial p_{\alpha'}}{\partial x^{\beta'}}} \qquad 29.3$$

The addition of $\partial\mathbf{p}'/\partial\theta$ indicates that $\Delta p\theta$ has to be replaced by $\Delta\theta$ by multiplying its column by p. The equation of hunting of the new system is

$$\mathbf{P}' = \mathbf{Z}' \cdot \Delta\mathbf{v}' \qquad 29.4$$

where

$$\Delta\mathbf{v} = \begin{bmatrix} \Delta i & \Delta\theta \end{bmatrix} \quad \text{while} \quad \Delta\dot{\mathbf{x}} = \begin{bmatrix} \Delta i & \Delta(p\theta) \end{bmatrix} \qquad 29.5$$

Synchronous Machine Connected to Infinite Bus

(*a*) When a synchronous machine is connected to an infinite bus (Fig. 29.1), its $\mathbf{e}_g$ (equation 18.17) is

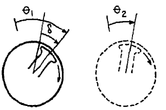


FIG. 29.1 Alternator connected to infinite bus.

$$\mathbf{e}_g = \begin{array}{ccccc} d_f & d_k & d_a & q_a & q_f \\ \hline E & & e\sin\delta & e\cos\delta & 0 \end{array} \qquad 29.6$$

where $\delta = \theta_1 - \theta_2 = \theta_{\text{alt.}} - \theta_{\text{bus}}$. Since its $\mathbf{C}$ is the unit tensor, therefore $\mathbf{e}_g' = \mathbf{e}_g$ and

$$\Delta\mathbf{e}_g' = \frac{\partial\mathbf{e}_g'}{\partial\delta}\Delta\delta = \frac{\partial\mathbf{e}_g'}{\partial\theta_1}\Delta\theta_1 - \frac{\partial\mathbf{e}_g'}{\partial\theta_2}\Delta\theta_2 \qquad 29.7$$

Since $\Delta\theta_2 = 0$ (that is, since the infinite bus does not hunt), $\Delta\theta_1$ *can be replaced everywhere by* $\Delta\delta = \Delta\theta_1 - \Delta\theta_2$. Hence

$$\Delta\mathbf{e}_g' = \frac{\partial\mathbf{e}_g'}{\partial\theta_1}\Delta\theta_1 = \frac{\partial\mathbf{e}_g'}{\partial\delta}\Delta\delta = \begin{array}{ccccc} d_f & d_k & d_a & q_a & q_k \\ \hline & & e\cos\delta\,\Delta\delta & -e\sin\delta\,\Delta\delta & \end{array}$$

29.8

These voltage changes appear on the terminals in all cases, in addition to any outside voltage and torque changes P_g' that may be applied.

(*b*) Hence, by the law of transformation of $\mathbf{Z}_g$,

$$\mathbf{Z}_g' = \mathbf{C}_t\cdot\mathbf{Z}_g\cdot\mathbf{C} - \frac{\partial\mathbf{e}_g'}{\partial\delta} \qquad 29.9$$

$$\mathbf{Z}_g' = \begin{array}{c|cccccc} & d_f & d_k & d_a & q_a & q_k & s \\ \hline d_f & -r_{fd}-L_{fd}p & -M_{fk}p & -M_{fd}p & & & \\ d_k & -M_{fk}p & -r_{kd}-L_{kd}p & -M_{kd}p & & & \\ d_a & -M_{fd}p & -M_{kd}p & -r_a-L_{ad}p & L_{aq}p\theta & M_{kq}p\theta & B_dp-e\cos\delta \\ q_a & -M_{fd}p\theta & -M_{kd}p\theta & -L_{ad}p\theta & -r_a-L_{aq}p & -M_{kq}p & B_qp+e\sin\delta \\ q_k & & & & -M_{kq}p & -r_{kq}-L_{kq}p & \\ s & -i^{aq}M_{fd} & -i^{aq}M_{kd} & -i^{aq}L_{ad}+B_d & i^{ad}L_{aq}+B_q & i^{ad}M_{kq} & Mp^2 \end{array}$$

29.10

(Note that the first five rows and columns are $\mathbf{Z}$ of equation 16.31.)

$$\mathbf{P}'_g = \begin{array}{cccccc} \mathbf{d}_f & \mathbf{d}_k & \mathbf{d}_a & \mathbf{q}_a & \mathbf{q}_k & \mathbf{s} \\ \hline \Delta E & & \Delta e_d & \Delta e_q & & \Delta T \end{array}$$

$$\Delta\mathbf{v}' = \begin{array}{cccccc} \mathbf{d}_f & \mathbf{d}_k & \mathbf{d}_a & \mathbf{q}_a & \mathbf{q}_k & \mathbf{s} \\ \hline \Delta i^{fd} & \Delta i^{kd} & \Delta i^{ad} & \Delta i^{aq} & \Delta i^{kq} & \Delta\delta \end{array}$$

where $p\theta$ is written for $p\theta_1$ and $\delta = \theta_1 - \theta_2$.

By convention, central-station engineers use *generated* voltages and *impressed* torques. Hence the last equation of $\mathbf{Z}'_g$ has to be multiplied by -1 to correspond to this convention.

Elimination of Field Axes

If the field axes $\mathbf{d}_f$, $\mathbf{d}_k$, and $\mathbf{q}_k$ are eliminated by $\mathbf{Z}' = \mathbf{Z}_4 - \mathbf{Z}_3 \cdot \mathbf{Z}_1^{-1} \cdot \mathbf{Z}_2$ and $\mathbf{e}' = \mathbf{e}_2 - \mathbf{Z}_3 \cdot \mathbf{Z}_1^{-1} \cdot \mathbf{e}_1$, the simplified equations are

$$\Delta\mathbf{v} = \begin{array}{ccc} \mathbf{d}_a & \mathbf{q}_a & \mathbf{s} \\ \hline \Delta i^{ad} & \Delta i^{aq} & \Delta\delta \end{array}$$

$$\mathbf{Z}''_g = \begin{array}{c|ccc|} & \mathbf{d}_a & \mathbf{q}_a & \mathbf{s} \\ \hline \mathbf{d}_a & -r_a - L_d(p)p & L_q(p)p\theta & B_d p - e\cos\delta \\ \mathbf{q}_a & -L_d(p)p\theta & -r_a - L_q(p)p & B_q p + e\sin\delta \\ \mathbf{s} & -i^q L_d(p) + B_d & i^d L_q(p) + B_q & Mp^2 \\ \hline \end{array} \qquad \mathbf{P}''_g = \begin{array}{|c|} \hline \Delta e_d - G(p)p\Delta E \\ \Delta e_q - G(p)p\theta\Delta E \\ \Delta T - i^q G(p)\Delta E \\ \hline \end{array} \qquad 29.11$$

During steady hunting, $p = jh\omega$ and

$$\mathbf{Z}''_s = \begin{array}{c|ccc|} & \mathbf{d}_a & \mathbf{q}_a & \mathbf{s} \\ \hline \mathbf{d}_a & -r_a - jhx_d(jh) & x_q(jh)p\theta & jhB_d - e\cos\delta \\ \mathbf{q}_a & -x_d(jh)p\theta & -r_a - jhx_d(jh) & jhB_q + e\sin\delta \\ \mathbf{s} & -i^q x_d(jh) + B_d & i^d x_q(jh) + B_q & Mp^2 \\ \hline \end{array} \qquad \mathbf{P}''_g = \begin{array}{|c|} \hline \Delta e_d - G(jh)jh\Delta E \\ \Delta e_q - G(jh)p\theta\Delta E \\ \Delta T - i_q G(jh)\Delta E \\ \hline \end{array} \qquad 29.12$$

(*b*) Using the per unit symbols of the central-station engineers

$L_d(p) = x_d(p)$	$r + L_d(p)p = z_d(p)$	$B_d = -\psi_{q0}$	$i^d = i_{d0}$
$L_q(p) = x_q(p)$	$r + L_q(p)p = z_q(p)$	$B_q = \psi_{d0}$	$i^q = i_{q0}$

$$\Delta \mathbf{v}'' = \begin{array}{ccc} \text{d} & \text{q} & \text{s} \\ \hline \Delta i_d & \Delta i_q & \Delta\delta \end{array}$$

$$\mathbf{Z}_g'' = \begin{array}{c|ccc} & \text{d} & \text{q} & \text{s} \\ \hline \text{d} & -z_d(p) & x_q(p)p\theta & -\psi_{q0}p - e\cos\delta \\ \text{q} & -x_d(p)p\theta & -z_q(p) & \psi_{d0}p + e\sin\delta \\ \text{s} & -i_{q0}x_d(p) - \psi_{q0} & i_{d0}x_q(p) + \psi_{d0} & Mp^2 \end{array} \qquad 29.13$$

$$\mathbf{P}_g'' = \begin{array}{c|c} \text{d} & \Delta e_d - pG(p)\Delta E \\ \text{q} & \Delta e_q - p\theta G(p)\Delta E \\ \text{s} & \Delta T - i_{q0}G(p)\Delta E \end{array}$$

$x_d(p)$, $x_q(p)$, and $G(p)$ are defined in equation 20.17 and ψ in 20.29.

It should be noted that the flux densities B occurring in the geometrical axes s may be replaced by flux linkages ψ only in synchronous and induction machines. In commutator machines (d-c. or a-c.), B and ψ are two different concepts with no apparent relation between them.

Numerical Example

Let a synchronous machine without excitation (or a polyphase induction motor running at synchronous speed) have the constants (when

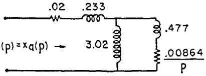


FIG. 29.2.

the rotor is connected to an impedance load) as shown in Fig. 29.2.

1. Steady-State Performance

$r_a = 0.02 \qquad e_a = 1.05 \qquad p\theta = \omega = 1$

$x_d = x_q = 3.02 + 0.233 = 3.253 \qquad \delta = 0$

By equation 20.33

$$\begin{array}{c} \\ \begin{vmatrix} i^d \\ \hline i^q \end{vmatrix} \end{array} = \begin{array}{c} \begin{vmatrix} e_a \sin\delta & e_a \cos\delta \end{vmatrix} \\ \begin{vmatrix} -r_a & x_d \\ -x_d & -r_a \end{vmatrix} \end{array} = \begin{array}{c} \begin{vmatrix} 0 & 1.05 \end{vmatrix} \\ \begin{vmatrix} -0.02 & 3.253 \\ -3.253 & -.02 \end{vmatrix} \end{array} = \begin{vmatrix} -0.324 \\ -0.00198 \end{vmatrix} \qquad 29.14$$

$$B_d = i^q x_s = -0.00198 \times 3.253 = -0.0064$$

$$B_q = -i^d x_s = 0.324 \times 3.253 = 1.052$$

2. Hunting Performance

$$x_d(jh) = x_q(jh) = 0.233 + \frac{1}{\dfrac{1}{3.02} + \dfrac{1}{0.477 - j0.216}} = 0.656 - j0.161$$

$$h = 0.04 \quad jhx_d(jh) = 0.00645 + j0.0262$$

Substituting into equation 29.13 (ignoring Mp^2)

$$\mathbf{Z}_g = \begin{vmatrix} -0.02645 - j0.0262 & 0.656 - j0.161 & -1.05 + j0.000256 \\ -0.656 + j0.161 & -0.02645 - j0.0262 & j0.042 \\ -0.0051 - j0.000319 & 0.84 + j0.0522 & 0 \end{vmatrix} = \begin{vmatrix} Z_1 & Z_2 \\ Z_3 & 0 \end{vmatrix} \qquad 29.15$$

$$\Delta T = -Z_3 \cdot Z_1^{-1} \cdot Z_2 = 1.26 + j0.394 \qquad 29.16$$

Therefore

$$T_S = 1.26$$

$$T_D = \frac{0.394}{0.04} = 9.84 \qquad 29.17$$

The system is stable at the frequency of hunting $0.04 \times 60 = 2.4$ cycles per second.

Double-Fed Induction Motor

When the impressed voltage vector of a polyphase induction motor is

$$\mathbf{e} = \mathbf{p} = \begin{array}{c} \begin{array}{ccccc} \mathrm{a}_s & \mathrm{a}_r & \mathrm{b}_r & \mathrm{b}_s & \mathrm{s} \end{array} \\ \begin{vmatrix} & -e_3 \sin\delta & e_3 \cos\delta & e_1 & \end{vmatrix} \end{array} \qquad 29.18$$

where $\delta = \theta_2 + \theta_3 - \theta_1$ (and $e = i^f Mp\theta$), then $\mathbf{Z}'$ of equation 27.18 has to be supplemented by $-\partial \mathbf{p}/\partial \delta$

$$\Delta \mathbf{p} = \frac{\partial \mathbf{p}}{\partial \theta_2} \Delta \theta_2 = \frac{\partial \mathbf{p}}{\partial \theta_2} \Delta \delta$$

since $\Delta\theta_1$ and $\Delta\theta_3$ are both zero. Hence

$$-\frac{\partial \mathbf{p}}{\partial \theta_2} = \begin{array}{ccccc} \mathbf{a}_s & \mathbf{a}_r & \mathbf{b}_r & \mathbf{b}_s & \mathbf{s} \\ \hline & e_3 \cos \delta & e_3 \sin \delta & & \end{array} \qquad 29.19$$

Multiplying the last column of equation 28.18 by p (thereby assuming $\Delta\delta$ as the variable in place of $\Delta p\delta$) and adding to it the above equation, the $\mathbf{Z}'$ of the double-fed induction motor is (assuming both reference axes fixed to the stator flux)

$$\mathbf{Z}' = \begin{array}{c|c|c|c|c|c} & \mathbf{a}_s & \mathbf{a}_r & \mathbf{b}_r & \mathbf{b}_s & \mathbf{s} \\ \hline \mathbf{a}_s & r_s + L_s p & Mp & -Mp\theta_1 & -L_s p\theta_1 & 0 \\ \mathbf{a}_r & Mp & r_r + L_r p & -L_r p\theta_s & -Mp\theta_s & (Mi^{bs} + L_r i^{br})p + e_3 \cos \delta \\ \mathbf{b}_r & Mp\theta_s & L_r p\theta_s & r_r + L_r p & Mp & -(Mi^{as} + L_r i^{ar})p + e_3 \sin \delta \\ \mathbf{b}_s & L_s p\theta_1 & Mp\theta_1 & Mp & r_s + L_s p & 0 \\ \mathbf{s} & i^{br}M & -i^{bs}M & i^{as}M & -i^{ar}M & Lp^2 \end{array}$$

29.20

where $e_3 = i^{f3}M_3 p\theta_3$ and $p\theta_s = p\theta_1 - p\theta_2$. The steady-state currents and voltages are all constant. During hunting $p = jh\omega$ and *

$$\mathbf{Z}' = \begin{array}{c|c|c|c|c|c} & \mathbf{a}_s & \mathbf{a}_r & \mathbf{b}_r & \mathbf{b}_s & \mathbf{s} \\ \hline \mathbf{a}_s & r_s + jhX_s & jhX_m & -X_m & -X_s & \\ \mathbf{a}_r & jhX_m & r_r + jhX_r & -sX_r & -sX_m & jh(X_r i^{br} + X_m i^{bs}) + e_3 \cos \delta \\ \mathbf{b}_r & sX_m & sX_r & r_r + jhX_r & jhX_m & -jh(X_r i^{ar} + X_m i^{as}) + e_3 \sin \delta \\ \mathbf{b}_s & X_s & X_m & jhX_m & r_s + jhX_s & \\ \mathbf{s} & i^{br}X_m & -i^{bs}X_m & i^{as}X_m & -i^{ar}X_m & Lp^2 \end{array}$$

29.21

The Hunting of Polyphase Machines

(*a*) The $\mathbf{Z}$ of interconnected systems may be established in two different manners:

* *A.T.E.M.*, p. 119.

1. The $\mathbf{Z}$ of the primitive (or other system) is transformed by $\mathbf{C}$. (The general laws of transformation, valid for the most general cases, are given in the next chapter.)

2. The transient pre-hunting equations $\mathbf{e}' = \mathbf{Z}' \cdot \mathbf{i}'$ and $T' = \mathbf{i}' \cdot \mathbf{G}' \cdot \mathbf{i}'$ are first established with the aid of $\mathbf{C}$, then small changes are made. (Or the scheme of equation 28.9 is established if the axes are stationary.)

The two methods serve as checks upon the correctness of the equations.

(b) When polyphase machines with smooth airgaps are interconnected, the matrices of equation 28.9 assume the form

$$\begin{array}{|c|} \hline \Delta \mathbf{e} \\ \hline \Delta T \\ \hline \end{array} = \begin{array}{|c|c|} \hline \Delta \mathbf{i}' & \Delta \theta \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \mathbf{Z}' & (\mathbf{G}' \cdot \mathbf{i}')p - \partial \mathbf{e}'/\partial \theta \\ \hline -\mathbf{i}'^{*} \cdot (\mathbf{G}'_s + \mathbf{G}'^{*}_{st}) & \mathbf{M}p^2 \\ \hline \end{array} \qquad 29.22$$

(Note that $\mathbf{G}' = \mathbf{G}'_s + \mathbf{G}'_r$, equation 16.33.)

Sometimes it is advantageous to establish $\mathbf{e}' = \mathbf{Z}' \cdot \mathbf{i}'$ and $T = \mathbf{i}' \cdot \mathbf{G}' \cdot \mathbf{i}'$ as polyphase (complex) equations in the manner of Chapter 22, afterward to change them into real form by equations 22.5 and 22.6. Then the hunting equations are established.

It is possible to establish $\mathbf{Z}'$ of any polyphase system by transforming $\mathbf{Z}$ of the primitive polyphase machine with the aid of a complex $\mathbf{C}$. That transformation is not considered here, however.

EXERCISES

1. Eliminate the stator axes and currents in equation 29.19.
2. For the double-fed induction motor:

(a) Find $\mathbf{C}$ in which the rotor axes $\mathbf{a}_r$ and $\mathbf{b}_r$ rotate with the rotor flux instead of the stator flux (the rotor flux is at an angle $(\theta + \theta_r)$ from a stationary axis, while the stator flux is at an angle $\theta_1 = \theta_s$ on Fig. 24.5).

(b) Establish the corresponding $\mathbf{Z}$.

(c) Find $\mathbf{C}$ in which both stator and rotor axes rotate with the rotor flux.

(d) Establish the corresponding $\mathbf{Z}$.

CHAPTER 30

THE LAW OF TRANSFORMATION OF $\mathbf{Z}$

The Classification of C

When the components of $\mathbf{C}$ are constants, the laws of transformation of $\mathbf{Z}$ and $\mathbf{Z}$ are $\mathbf{C}_t^*\cdot\mathbf{Z}\cdot\mathbf{C}$ and $\mathbf{C}_t^*\cdot\mathbf{Z}\cdot\mathbf{C} - \partial\mathbf{e}'/\partial\theta$.

The components of $\mathbf{C}$ may be functions of time that do not change because of hunting. This was true in equation 28.17 where the reference frame rotated *uniformly* with a velocity $p\theta$ with respect to the rotor no matter what the rotor itself did. (That is, $\Delta\mathbf{C} = 0$ but $p\mathbf{C} \neq 0$.) In that case, the laws of transformation of $\mathbf{Z}$ and $\mathbf{Z}$ are either as above, where the p in $\mathbf{Z}$ refers to $\mathbf{C}$ and $\mathbf{i}$, or equation 23.3 and

$$\boxed{\mathbf{Z}' = \mathbf{C}_t^*\cdot\mathbf{Z}\cdot\mathbf{C} + \mathbf{C}_t^*\cdot\mathbf{L}\cdot\frac{\partial\mathbf{C}}{\partial\theta}p\theta - \frac{\partial\mathbf{e}'}{\partial\theta}} \qquad 30.1$$

where p refers only to $\mathbf{i}$.

When two interconnected synchronous machines run at a constant angle δ, then during hunting this δ (occurring in their $\mathbf{C}$, equation 20.41) also varies. Then $\Delta\mathbf{C}$ is not zero even though $p\mathbf{C}$ is zero.

The law of transformation of $\mathbf{Z}$ when $\Delta\mathbf{C} \neq 0$ is to be investigated now.

The Laws of Transformation of $\Delta\dot{x}^\alpha$ and Δp_α

It has been shown that the laws of transformation of the velocity vector $\dot{x}^\alpha$ and the force vector p_α are those of tensors, namely,

$$\dot{\mathbf{x}} = \mathbf{C}\cdot\dot{\mathbf{x}}' \qquad \begin{aligned}\dot{x}^\alpha &= C^\alpha_{\alpha'}\dot{x}^{\alpha'}\\ p_\alpha &= C^{\alpha'}_\alpha p_{\alpha'}\end{aligned} \qquad 30.2$$

The question now arises: What are the laws of transformation of their differentials $d\dot{x}^\alpha$ and dp_α (or $\Delta\dot{x}^\alpha$ and Δp_α)? If the components of $C^\alpha_{\alpha'}$ are constants, they transform as $\dot{x}^\alpha$ and p_α; but if $C^\alpha_{\alpha'}$ is a function of the variables or the parameters, then their laws of transformation are more complicated.

Making small changes in the above equations

$$\boxed{\Delta\dot{\mathbf{x}} = \mathbf{C}\cdot\Delta\dot{\mathbf{x}}' + \Delta\mathbf{C}\cdot\dot{\mathbf{x}}'} \qquad \boxed{\Delta\dot{x}^{\alpha} = C^{\alpha}_{\alpha'}\Delta\dot{x}^{\alpha'} + \Delta C^{\alpha}_{\alpha'}\dot{x}^{\alpha'}} \tag{30.3}$$

$$\boxed{\Delta\mathbf{p} = \mathbf{C}_t^{-1}\cdot\Delta\mathbf{p}' + \Delta\mathbf{C}_t^{-1}\cdot\mathbf{p}'} \qquad \boxed{\Delta p_{\alpha} = C^{\alpha'}_{\alpha}\Delta p_{\alpha'} + \Delta C^{\alpha'}_{\alpha}p_{\alpha'}} \tag{30.4}$$

where

$$\Delta\mathbf{C} = \frac{\partial\mathbf{C}}{\partial\boldsymbol{\theta}}\,\Delta\boldsymbol{\theta} \qquad \Delta C^{\alpha}_{\alpha'} = \frac{\partial C^{\alpha}_{\alpha'}}{\partial\theta^{\beta}}\,\Delta\theta^{\beta} \tag{30.5}$$

In the extra term of the law of transformation of $\Delta\dot{\mathbf{x}}$ *there occurs* $\dot{\mathbf{x}}$, *and in that of* $\Delta\mathbf{p}$ *occurs* $\mathbf{p}$. Hence $\Delta\dot{x}^{\alpha}$ and Δp_{α} are neither tensors nor geometric objects but "partial geometric objects" since, in their law of transformation, not only C^{α}_{α} and $L_{\alpha\beta}$ but also $\dot{x}^{\alpha}$ (and p_{α}) occur.

Since in the general case neither $\mathcal{Z}$, nor $\Delta\dot{\mathbf{x}}$ nor $\Delta\mathbf{p}$ are tensors, the equation of hunting $\Delta\mathbf{p} = \mathcal{Z}\cdot\Delta\dot{\mathbf{x}}$ is no longer a tensor equation.

The Law of Transformation of $\mathcal{Z}_{\alpha\beta}$

When $\mathbf{C}$ is a function of a parameter δ (such as the angle between two synchronous machines running at the same speed), $\mathcal{Z}$ is no more a tensor. For the primitive machine let

$$\Delta\mathbf{p} = \mathcal{Z}\cdot\Delta\dot{\mathbf{x}}$$

Substituting $\Delta\mathbf{p}$ and $\Delta\dot{\mathbf{x}}$ from equations 30.4 and 30.3,

$$\mathbf{C}_t^{-1}\cdot\Delta\mathbf{p}' + \Delta\mathbf{C}_t^{-1}\cdot\mathbf{p}' = \mathcal{Z}\cdot(\mathbf{C}\cdot\Delta\dot{\mathbf{x}}' + \Delta\mathbf{C}\cdot\dot{\mathbf{x}}')$$

Multiplying by $\mathbf{C}_t$,

$$\Delta\mathbf{p}' = \left[\mathbf{C}_t\cdot\mathcal{Z}\cdot\mathbf{C} + \mathbf{C}_t\cdot\mathcal{Z}\cdot\frac{\partial\mathbf{C}}{\partial\boldsymbol{\theta}}\cdot\dot{\mathbf{x}}' - \mathbf{C}_t\cdot\frac{\partial\mathbf{C}_t^{-1}}{\partial\boldsymbol{\theta}}\cdot\mathbf{p}'\right]\Delta\mathbf{v} \tag{30.6}$$

where $\Delta\mathbf{v}$ contains $\Delta\mathbf{i}$ and $\Delta\theta$ as its components.

Hence, when $\mathbf{C}$ is a function of a parameter, two terms are added to $\mathcal{Z}'$, one by the law of transformation of $\Delta\dot{\mathbf{x}}$, the other by that of $\Delta\mathbf{p}$.

Since $\Delta\mathbf{p}'$ is $(\partial\mathbf{p}/\partial\boldsymbol{\theta})\cdot\Delta\boldsymbol{\theta} + \mathbf{P}'$, the equation of hunting of the new machine is

$$\mathbf{P}' = \mathcal{Z}'\cdot\Delta\mathbf{v}' \tag{30.7}$$

where *the law of transformation of* $\mathcal{Z}'$ is

$$\mathcal{Z}' = \mathbf{C}_t\cdot\mathcal{Z}\cdot\mathbf{C} + \mathbf{C}_t\cdot\mathcal{Z}\cdot\frac{\partial\mathbf{C}}{\partial\boldsymbol{\theta}}\cdot\dot{\mathbf{x}}' - \mathbf{C}_t\cdot\frac{\partial\mathbf{C}_t^{-1}}{\partial\boldsymbol{\theta}}\cdot\mathbf{p}' - \frac{\partial\mathbf{e}'}{\partial\boldsymbol{\theta}} \tag{30.8}$$

When the Inverse of C Does Not Exist

(*a*) When $\mathbf{C}^{-1}$ does not exist (required in the last but one term) a new expression may be derived for it. Let

$$\mathbf{C}^{-1}\cdot\mathbf{C} = \mathbf{I} \qquad 30.9$$

Differentiating $\Delta(\mathbf{C}^{-1}\cdot\mathbf{C}) = 0 = \Delta\mathbf{C}^{-1}\cdot\mathbf{C} + \mathbf{C}^{-1}\cdot\Delta\mathbf{C}$. Hence

$$\mathbf{C}_t\cdot\Delta\mathbf{C}_t^{-1} = -\Delta\mathbf{C}_t\cdot\mathbf{C}_t^{-1} \qquad 30.10$$

Substituting, $\mathbf{C}_t\cdot\Delta\mathbf{C}_t^{-1}\cdot\mathbf{p}' = \Delta\mathbf{C}_t\cdot\mathbf{C}_t^{-1}\cdot\mathbf{p}' = -\Delta\mathbf{C}_t\cdot\mathbf{C}_t^{-1}\cdot\mathbf{C}_t\cdot\mathbf{p}$

or

$$\mathbf{C}_t\cdot\Delta\mathbf{C}_t^{-1}\cdot\mathbf{p}' = -\Delta\mathbf{C}_t\cdot\mathbf{p} = -\Delta\mathbf{C}_t\cdot\mathbf{e} \qquad 30.11$$

Hence $\mathbf{C}^{-1}$ disappears, but in its place appears $\mathbf{e}$, the applied voltage existing before hunting and before the transformation.

Since $\mathbf{Z}'$ contains the steady-state $\mathbf{i}_{0'}$, it is advantageous to express $\mathbf{e}$ also in terms of $\mathbf{i}_0'$ as

$$\mathbf{e} = \mathbf{Z}\cdot\mathbf{i} = \mathbf{Z}\cdot\mathbf{C}\cdot\mathbf{i}' \qquad 30.12$$

Hence the law of transformation of $\mathbf{Z}$ is

$$\boxed{\mathbf{Z}' = \mathbf{C}_t^*\cdot\mathbf{Z}\cdot\mathbf{C} + \mathbf{C}_t^*\cdot\mathbf{Z}\cdot\frac{\partial\mathbf{C}}{\partial\theta}\cdot\dot{\mathbf{x}}' + \frac{\partial\mathbf{C}_t^*}{\partial\theta}\cdot\mathbf{Z}\cdot\mathbf{C}\cdot\mathbf{i}' - \frac{\partial\mathbf{e}}{\partial\theta}} \qquad 30.13$$

$$Z_{\alpha'\beta'} = C_{\alpha'}^{\alpha}Z_{\alpha\beta}C_{\beta'}^{\beta} + C_{\alpha'}^{\alpha}Z_{\alpha\beta}\frac{\partial C_{\gamma'}^{\beta}}{\partial x^{\beta'}}\dot{x}^{\gamma'} + \frac{\partial C_{\alpha'}^{\alpha}}{\partial x^{\beta'}}Z_{\alpha\beta}C_{\gamma'}^{\beta'}i^{\gamma'} - \frac{\partial e_{\alpha'}}{\partial x^{\beta'}}$$

(*b*) When the components of $\mathbf{C}$ and $\mathbf{i}_0$ contain functions of time, then all p in $\mathbf{Z}$ refer to all such variables to the right of them. In such cases the order of the components in the multiplication cannot be changed. The expanded law of transformation for such conditions is given elsewhere.*

Mechanical Problems

It may be mentioned that in most mechanical problems (in holonomic dynamical systems) the equation of hunting is not $\Delta\mathbf{p} = \mathbf{Z}\Delta\dot{\mathbf{x}}$ but

$$\Delta\mathbf{p} = \mathbf{Z}\cdot\Delta\mathbf{x} \quad \Big| \quad \Delta p_\alpha = Z_{\alpha\beta}\Delta x^\beta \qquad 30.14$$

where $\mathbf{x}$ are the variables. Since the law of transformation of $\Delta\mathbf{x}$ is

$$\Delta\mathbf{x} = \mathbf{C}\cdot\Delta\mathbf{x}' \quad \Big| \quad \Delta x^\alpha = C_{\alpha'}^{\alpha}\Delta x^{\alpha'} \qquad 30.15$$

* *A.T.E.M.*, p. 128, equations 42 and 46.

the extra term in equation 30.13, $\mathbf{C}_t^* \cdot \mathbf{Z} \cdot (\partial \mathbf{c}/\partial \mathbf{x}) \cdot \dot{\mathbf{x}}'$ (due to the law of transformation of $\Delta\dot{\mathbf{x}}$), is absent; hence *in mechanical oscillation problems the law of transformation of* $\mathbf{Z}$ *is*

$$\mathbf{Z}' = \mathbf{C}_t^* \cdot \mathbf{Z} \cdot \mathbf{C} - \frac{\partial \mathbf{C}_t^*}{\partial \mathbf{x}} \cdot \mathbf{p} - \frac{\partial \mathbf{p}'}{\partial \mathbf{x}} \quad \Big| \quad Z_{\alpha'\beta'} = Z_{\alpha\beta} C_{\alpha'}^{\alpha} C_{\beta'}^{\beta} - \frac{\partial C_{\alpha'}^{\gamma'}}{\partial x^{\beta'}} p_{\gamma'} - \frac{\partial p_{\alpha'}}{\partial x^{\beta'}} \qquad 30.16$$

where $\mathbf{p} = p_\alpha$ is the steady-state force equation of the system *before* interconnection and $\mathbf{p}' = p_{\alpha'}$ is the applied steady-state force *after* the interconnection. All $p = d/dt$ in $\mathbf{Z}$ refer to both $\mathbf{C}$ and $\Delta\mathbf{x}$.

The equation of transformation 30.13 developed for electrical machinery is valid for *non-holonomic* dynamical (mechanical) systems in which the velocities also are subjected to small changes.

EXERCISES

1. Find the transient $\mathbf{Z}$ of the amplidyne of Fig. 18.3.
2. If the frequency of oscillation of the amplidyne is $h\omega$, what is the steady-state $\mathbf{Z}$?
3. If a synchronous machine (and the bus) run at a speed $v\omega$ and the field hunts at a frequency $h\omega$, what is the steady-state $\mathbf{Z}$ of the synchronous machine?
4. Using the design constants of Fig. 20.6, find T_S and T_D of the synchronous machine at the angles stated in exercise *e*, Chapter 20.

CHAPTER 31

THE EQUATION OF MOTION *

The Electromagnetic Field Tensor $F_{\alpha\beta}$

(*a*) The two equations completely determining the accelerated motion of a single rotating machine with relatively stationary axes have been given in equations 28.1 and 28.2 as

$$\begin{array}{l|l} \mathbf{e} = \mathbf{R}\cdot\mathbf{i} + \mathbf{L}\cdot p\mathbf{i} + p\theta\mathbf{G}\cdot\mathbf{i} & e_m = R_{mn}i^n + L_{mn}pi^n + p\theta G_{mn}i^n \\ T = Rp\theta + Mp^2\theta - \mathbf{i}\cdot\mathbf{G}\cdot\mathbf{i} & T = Mp^2\theta - G_{mn}i^m i^n \end{array} \qquad 31.1$$

or

$$\begin{array}{l|l} \mathbf{e} = \mathbf{R}\cdot\mathbf{i} + p\varphi + \mathbf{B}p\theta & e_m = R_{mn}i^n + p\varphi_m + B_m p\theta \\ T = Rv + p(mv) - \mathbf{i}\cdot\mathbf{B} & T = Mp^2\theta - i^n B_n \end{array} \qquad 31.2$$

where the mechanical friction R is introduced for the sake of symmetry.

(*b*) These two equations also can be expressed as one equation in terms of "compound" tensors (analogously to the equations of hunting) by introducing the geometrical axis s to express along it the mechanical quantities. That is, let the following compound tensors be introduced:

$$\mathbf{p} = p_\alpha = \begin{array}{|c|c|}\hline \mathbf{e} & T \\ \hline\end{array} \qquad \mathbf{x} = \dot{\mathbf{x}}^\alpha = \begin{array}{|c|c|}\hline \mathbf{i} & v \\ \hline\end{array} \qquad \mathbf{a} = a_{\alpha\beta} = \begin{array}{|c|c|}\hline \mathbf{L} & \\ \hline & M \\ \hline\end{array} \qquad \mathbf{r} = r_{\alpha\beta} = \begin{array}{|c|c|}\hline \mathbf{R} & \\ \hline & R \\ \hline\end{array} \qquad 31.3$$

The tensor $a_{\alpha\beta}$ is called the "metric tensor."

It should be expressly noted that *the rotor flux-density vector* **B** *occurs twice in the complete set,* in particular:

1. In the voltage equation it produces generated voltages.
2. In the torque equation (with negative sign) it produces torque.

The **B** vector has to be arranged as a *skew-symmetric tensor* of valence 2, $\mathbf{F} = F_{\alpha\beta}$

$$\mathbf{F} = F_{\alpha\beta} = \begin{array}{c|c|c|} \multicolumn{1}{c}{\alpha\backslash\beta} & \multicolumn{2}{c}{} \\ \hline & & \mathbf{B} \\ \hline & -\mathbf{B} & \\ \hline \end{array} = \begin{array}{c|c|c|} \multicolumn{1}{c}{\alpha\backslash\beta} & \multicolumn{2}{c}{} \\ \hline & & \mathbf{G}\cdot\mathbf{i} \\ \hline & -\mathbf{i}\cdot\mathbf{G}_t & \\ \hline \end{array} \qquad 31.4$$

* *A.T.E.M.*, p. 95.

the so-called "electromagnetic field tensor" that occurs in the tensorial field equations of Maxwell, equation 14.7. (It should be noted that **F** also occurs as a component of **Z**, equation 28.9.)

(*c*) In terms of these compound tensors, the two equations can be combined into one, the so-called equation of motion

$$\mathbf{p} = \mathbf{r}\cdot\dot{\mathbf{x}} + \mathbf{a}\cdot p\dot{\mathbf{x}} + \mathbf{F}\cdot\dot{\mathbf{x}} \quad \Big| \quad p_\alpha = r_{\alpha\beta}\dot{x}^\beta + a_{\alpha\beta}\dot{x}^\beta + F_{\alpha\beta}\dot{x}^\beta \qquad 31.5$$

where the field tensor **F** is a function of **i**. *In this form the equation may include any number of pθ's, not only one.*

Acceleration of Direct-Current Machines

In many d-c. machine applications it may be assumed that *during acceleration the rotor flux-density* B *remains constant along each axis.* Then the field tensor F is constant and $\dot{\mathbf{x}}$ may be factored out as

$$\mathbf{p} = (\mathbf{r} + \mathbf{a}p + \mathbf{F})\cdot\dot{\mathbf{x}} = \mathbf{Z}_a\cdot\dot{\mathbf{x}} \qquad 31.6$$

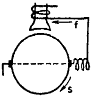


FIG. 31.1. Compound d-c. machine.

This is a set of linear differential equations with constant coefficients, just as $\mathbf{e} = \mathbf{Z}\cdot\mathbf{i}$ or $\Delta\mathbf{p} = \mathbf{Z}\cdot\Delta\mathbf{i}$ are, and *can be solved with Heaviside's expansion theorem for the instantaneous velocity and currents* $\dot{\mathbf{x}}$ *as* $\dot{\mathbf{x}} = \mathbf{Z}_a^{-1}\cdot\mathbf{p}$.

As an example, let the acceleration of the compound machine of Fig. 31.1 be analyzed. The **Z** of its primitive machine is

$$\mathbf{Z} = \begin{array}{c|cccc} & \mathrm{d}_s & \mathrm{q}_r & \mathrm{q}_s & \mathrm{s} \\ \hline \mathrm{d}_s & r_{ds} + L_{ds}p & & & \\ \mathrm{q}_r & -M_d p\theta & r_r + L_{qr}p & & \\ \mathrm{q}_s & & M_q p & r_{qs} + L_{qs}p & \\ \mathrm{s} & & & & Mp \end{array} \qquad \mathbf{C} = \begin{array}{c|cc} & \mathrm{f} & \mathrm{s} \\ \hline \mathrm{d}_s & n_d & \\ \mathrm{q}_r & 1 & \\ \mathrm{q}_s & n_q & \\ \mathrm{s} & & 1 \end{array} \qquad 31.7$$

$\mathbf{Z}' = \mathbf{C}_t\cdot\mathbf{Z}\cdot\mathbf{C}$, where $\mathbf{Z}'$ may be expressed as $\mathbf{r}' + \mathbf{a}'p$, so that

$$\mathbf{r}' = \begin{array}{c|cc} & \mathrm{f} & \mathrm{s} \\ \hline \mathrm{f} & r_r + n_d^2 r_{ds} + n_q^2 r_{qs} & \\ \mathrm{s} & & \end{array} \qquad \mathbf{a}' = \begin{array}{c|cc} & \mathrm{f} & \mathrm{s} \\ \hline \mathrm{f} & n_d^2 L_{ds} + L_{qr} + n_q^2 L_{qs} + 2n_q M_q & \\ \mathrm{s} & & M \end{array}$$

Since the flux-density vector is $B = -M_d n_d i^f$

$$\mathbf{F}' = \begin{array}{c|cc} & \mathrm{f} & \mathrm{s} \\ \hline \mathrm{f} & & -M_d n_d i^f \\ \mathrm{s} & M_d n_d i^f & \end{array}$$

$$\mathbf{p}' = \begin{array}{cc} \mathrm{f} & \mathrm{s} \\ \hline e_f 1 & T1 \end{array}$$

$$Z_a' = \begin{array}{c|cc} & \mathrm{f} & \mathrm{s} \\ \hline \mathrm{f} & \begin{array}{c} r_r + n_d^2 r_{ds} + n_q^2 r_{qs} \\ +(n_d^2 L_{ds} + L_{qr} + \\ n_q^2 L_{qs} + 2n_q M_q)p \end{array} & -M_d n_d i^f \\ \mathrm{s} & M_d n_d i^f & Mp \end{array} \qquad 31.8$$

The solution of $\dot{\mathbf{x}} = \mathbf{Z}_a^{-1}\cdot\mathbf{p}$ gives the instantaneous current i^f and rotor velocity v^s.

The Torsion Tensor $S_{\alpha\beta\gamma}$

(*a*) Instead of the two **B**'s let the two G's be arranged into a compound tensor. Since the **B**'s are arranged into a tensor of valence 2, the G's must be arranged into a tensor of valence 3 (Fig. 31.2) called the "torsion tensor" $T_{\alpha\beta\gamma}$. It is skew symmetric in the first and third indices. That is,

$$T_{\alpha\beta\gamma} = -T_{\gamma\beta\alpha} \qquad 31.9$$

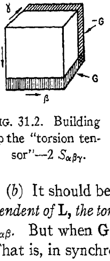


FIG. 31.2. Building up the "torsion tensor"—2 $S_{\alpha\beta\gamma}$.

Many writers call half of $T_{\alpha\beta\gamma}$ the torsion tensor $S_{\alpha\beta\gamma}$ so that

$$T_{\alpha\beta\gamma} = 2S_{\alpha\beta\gamma} \qquad 31.10$$

(*b*) It should be noted that, *in commutator machines where* G *is independent of* **L**, *the torsion tensor* $S_{\alpha\beta\gamma}$ *is also independent of the metric tensor* $a_{\alpha\beta}$. But when $\mathbf{G} = \mathbf{\Upsilon}_t\cdot\mathbf{L}$, then $S_{\alpha\beta\gamma}$ can be expressed in terms of $a_{\alpha\beta}$. That is, in synchronous and induction machines

$$S_{\alpha\beta\gamma} = \frac{1}{2}\, a_{\gamma\delta} C_\alpha^{\alpha'} C_\beta^{\beta'} \left(\frac{\partial C_{\beta'}^\delta}{\partial x^{\alpha'}} - \frac{\partial C_{\alpha'}^\delta}{\partial x^{\beta'}}\right) \qquad 31.11$$

where C_β^α is a function of the displacements x^α of the rotor conductors.

(*c*) In terms of $S_{\alpha\beta\gamma}$ the equation of motion (valid for machines with relatively stationary axes) becomes

$$p_\alpha = r_{\alpha\beta}\frac{dx^\beta}{dt} + a_{\alpha\beta}\frac{d^2x^\beta}{dt} + 2S_{\gamma\beta\alpha}\frac{dx^\gamma}{dt}\frac{dx^\beta}{dt} \qquad 31.12$$

where x^α represents the charges and instantaneous displacements. In the general case these differential equations can be solved only by step-by-step methods.

The Affine Connection $\Gamma_{\alpha\beta,\gamma}$

(a) When the reference frames rotate with any arbitrary velocity $p\theta'$, then the additional **V** that appears in equation 26.8 may also be incorporated with the two G's into a geometric object of valence 3 (Fig. 31.3), the so-called affine connection $\Gamma_{\alpha\beta,\gamma}$. (It is not a tensor, but a geometric object, since **V** is not a tensor. It is customary to place a comma before its last index.)

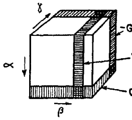


FIG. 31.3. Building up the "affine connection" $\Gamma_{\alpha\beta,\gamma}$.

In terms of the affine connection, the equation of motion is

$$p_\alpha = r_{\alpha\beta}\frac{dx^\beta}{dt} + a_{\alpha\beta}\frac{d^2x^\gamma}{dt^2} + \Gamma_{\beta\gamma,\alpha}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt} \qquad 31.13$$

This equation represents the performance of any number of machines with any type of rotating frame.

In the general case of commutator machines the components of $\Gamma_{\beta\gamma,\alpha}$ are arbitrary quantities independent of $a_{\alpha\beta}$. They represent the mutual inductances due to the existence of rotations of conductors and reference frames.

The law of transformation of $\Gamma_{\alpha\beta,\gamma}$ is analogous to that of **V**, equation 26.12

$$\Gamma_{\alpha'\beta,\gamma'} = \Gamma_{\alpha\beta,\gamma}C^\alpha_{\alpha'}C^\beta_{\beta'}C^\gamma_{\gamma'} + a_{\gamma\alpha}C^\gamma_{\gamma'}\frac{\partial C^\alpha_{\alpha'}}{\partial x^{\beta'}} \qquad 31.14$$

(b) If the parameter t (time) is replaced by s (distance), the resulting equation represents a line in an n-dimensional "*non-Riemannian*" space. Hence with t the equation of motion 31.13 may be said to represent the motion of a particle in an n-dimensional non-Riemannian space.

The Christoffel Symbol $[\alpha\beta,\gamma]$

(a) In special reference frames the components of $\Gamma_{\alpha\beta,\gamma}$ assume special forms. For instance, for the first primitive machine and in general for machines with relatively stationary axes

$$\Gamma_{\alpha\beta,\gamma} = 2S_{\alpha\beta\gamma} \qquad 31.15$$

Another very important special case is the holonomic frame of the

second primitive machine. The equations of voltage and torque of Maxwell are

$$\mathbf{e} = \mathbf{R}\cdot\mathbf{i} + p(\mathbf{L}\cdot\mathbf{i})$$
$$T = Rp\theta + Mpv - \frac{1}{2}\cdot\mathbf{i}\cdot\frac{\partial L}{\partial \theta}\cdot\mathbf{i} \qquad 31.16$$

The first equation can be written

$$\mathbf{e} = \mathbf{R}\cdot\mathbf{i} + \mathbf{L}p\mathbf{i} + p\theta\frac{\partial L}{\partial \theta}\cdot\mathbf{i}$$

If $-\frac{1}{2}\frac{\partial \mathbf{L}}{\partial \theta}$ and $\frac{\partial \mathbf{L}}{\partial \theta}$ are arranged analogously to the two **G**'s (Fig. 31.4*a*) the resultant is a geometric object of valence 3, the so-called holonomic Christoffel symbol.

It is customary (and from a tensorial point of view necessary) to divide $\partial\mathbf{L}/\partial\theta$ into the sum of two equal matrices $(\frac{1}{2})\partial\mathbf{L}/\partial\theta$ and arrange them as shown in Fig. 31.4*b*. The resultant equation of motion in both cases gives the same answer.

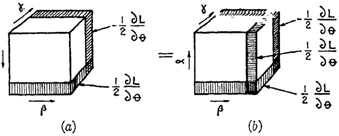


FIG. 31.4. Building up the "Christoffel symbol" $[\alpha\beta,\gamma]$.

(*b*) Since in such holonomic frames

$$\Gamma_{\alpha\beta,\gamma} = [\alpha\beta,\gamma] \qquad 31.17$$

the equation of motion for holonomic reference frames is

$$p_\alpha = r_{\alpha\beta}\frac{dx^\beta}{dt} + a_{\alpha\beta}\frac{d^2x^\beta}{dt^2} + [\beta\gamma,\alpha]\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt} \qquad 31.18$$

where the Christoffel symbol is defined (with any number of rotating members) in terms of the metric tensor $a_{\alpha\beta}$ as

$$[\alpha\beta,\gamma] = \frac{1}{2}\left(\frac{\partial a_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial a_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial a_{\alpha\beta}}{\partial x^\gamma}\right) \qquad 31.19$$

Its law of transformation is the same as that of $\Gamma_{\alpha\beta,\gamma}$, namely, equation 31.14.

(*c*) Note that: (1) The order of the indices in the denominator is α, β, γ the same as in $[\alpha\beta,\gamma]$; (2) the two indices in each numerator differ from those in their respective denominators.

The first two terms give the generated voltages and the last term (the negative) gives the torque.

Along non-holonomic reference frames $[\alpha\beta,\gamma]$ has a more complex form.

(*d*) The above equation of motion is said to represent the motion of a particle in an n-dimensional *Riemannian* space.

The Dynamical Equation of Lagrange *

It will be proved that the *equation of motion for holonomic axes (equation 31.18) represents the well-known dynamical equation of Lagrange in an explicit form.* That is, the kinetic energy T of the dynamical equation is here replaced by its value $(1/2)a_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$.

(*a*) Starting with the equation of Lagrange,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^\gamma}\right) - \frac{\partial T}{\partial x^\gamma} + \frac{\partial F}{\partial \dot{x}^\gamma} = p_\gamma \qquad 31.20$$

let $T = (1/2)a_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$ and $F = (1/2)r_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$

$$\frac{\partial T}{\partial \dot{x}^\gamma} = \frac{1}{2}\frac{\partial(a_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta)}{\partial \dot{x}^\gamma} = \frac{1}{2}a_{\gamma\beta}\dot{x}^\beta + \frac{1}{2}a_{\alpha\gamma}\dot{x}^\alpha$$

This result is found by differentiating each tensor separately. Now $\partial a_{\alpha\beta}/\partial\dot{x}^\gamma = 0$, $\partial\dot{x}^\alpha/\partial\dot{x}^\gamma = \delta^\alpha_\gamma$ = unit tensor, and $a_{\alpha\beta}\delta^\alpha_\gamma = a_{\gamma\beta}(a_{\alpha\beta}$ is a function of x^γ but not of $\dot{x}^\gamma$).

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^\gamma}\right) = \frac{1}{2}\left(\frac{\partial a_{\gamma\beta}}{\partial x^\alpha}\frac{dx^\alpha}{dt}\dot{x}^\beta + a_{\gamma\beta}\frac{d^2x^\beta}{dt^2} + \frac{\partial a_{\alpha\gamma}}{\partial x^\beta}\frac{dx^\beta}{dt}\dot{x}^\alpha + a_{\alpha\gamma}\frac{d^2x^\alpha}{dt^2}\right)$$

$$= \frac{1}{2}\left(\frac{\partial a_{\gamma\beta}}{\partial x^\alpha} + \frac{\partial a_{\alpha\gamma}}{\partial x^\beta}\right)\dot{x}^\alpha\dot{x}^\beta + a_{\gamma\beta}\frac{d^2x^\beta}{dt^2}$$

$$\frac{\partial T}{\partial x^\gamma} = \frac{1}{2}\frac{\partial a_{\alpha\beta}}{\partial x^\gamma}\dot{x}^\alpha\dot{x}^\beta$$

$$\frac{\partial F}{\partial \dot{x}^\gamma} = \frac{1}{2}r_{\gamma\beta}\dot{x}^\beta + \frac{1}{2}r_{\alpha\gamma}\dot{x}^\alpha = r_{\gamma\beta}\dot{x}^\beta$$

Substituting into the equation of Lagrange, the *explicit* form of the equation of Lagrange comes out as

$$a_{\gamma\beta}\frac{d^2x^\beta}{dt^2} + \frac{1}{2}\left(\frac{\partial a_{\gamma\beta}}{\partial x^\alpha} + \frac{\partial a_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial a_{\alpha\beta}}{\partial x^\gamma}\right)\dot{x}^\alpha\dot{x}^\beta + r_{\gamma\beta}\dot{x}^\beta = p_\gamma \qquad 31.21$$

* Kron, "Quasi-Holonomic Dynamical Systems," *Physics*, vol. 7, April, 1936.

or

$$p_\gamma = r_{\gamma\beta}\dot{x}^\beta + a_{\gamma\beta}\frac{d\dot{x}^\beta}{dt} + [\alpha\beta,\gamma]\dot{x}^\alpha\dot{x}^\beta \tag{31.22}$$

(*b*) In the *general* case of commutator machines when $\Gamma_{\alpha\beta,\gamma}$ is not a function of $a_{\alpha\beta}$ (that is, when **G** is independent of **L**), *the equation of motion cannot be expressed in terms of the kinetic energy T* and it cannot be considered a modification of the Lagrangian equation. Classical dynamics has no equivalent concepts to offer, and the concepts of relativistic electrodynamics must be resorted to, from which the entities $\Gamma_{\alpha\beta,\gamma}$ and $S_{\alpha\beta,\gamma}$ have been borrowed. Classical dynamics employs only the Christoffel symbol $[\alpha\beta,\gamma]$.

In the *special* case of synchronous and induction machines when $\mathbf{G} = \mathbf{Y}_t \cdot \mathbf{L}$, then $\Gamma_{\alpha\beta,\gamma}$ can be expressed in terms of the kinetic energy T by using the Boltzmann-Hamel extension of the Lagrangian equation that has been developed for non-holonomic reference frames, namely

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^{\gamma'}}\right) - \frac{\partial T}{\partial x^{\gamma'}} + \frac{\partial T}{\partial \dot{x}^{\delta'}} C^\gamma_{\gamma'} C^\beta_{\beta'} \left(\frac{\partial C^{\delta'}_\gamma}{\partial x^\beta} - \frac{\partial C^{\delta'}_\beta}{\partial x^\gamma}\right)\dot{x}^{\beta'} + \frac{\partial F}{\partial \dot{x}^{\gamma'}} = f_{\gamma'} \tag{31.23}$$

The derivation is to be found in other publications.*.

EXERCISES

1. Find $\mathbf{Z}_a$ of the machines of Figs. 31.5 and 31.6.

2. Starting with the law of transformation of the metric tensor $a_{\alpha\beta}$, derive that of $\partial a_{\alpha\beta}/\partial x^\gamma$.

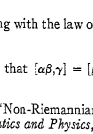


FIG. 31.5.

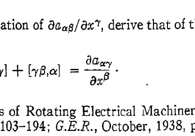


FIG. 31.6.

3. Starting with the law of transformation of $\partial a_{\alpha\beta}/\partial x^\gamma$, derive that of the Christoffel symbol.

4. Prove that $[\alpha\beta,\gamma] = [\beta\alpha,\gamma]$; $[\alpha\beta,\gamma] + [\gamma\beta,\alpha] = \dfrac{\partial a_{\alpha\gamma}}{\partial x^\beta}$.

* Kron, "Non-Riemannian Dynamics of Rotating Electrical Machinery," *Journal of Mathematics and Physics*, 1934, pp. 103–194; *G.E.R.*, October, 1938, p. 448.

CHAPTER 32

THE THIRD GENERALIZATION POSTULATE

Vectors without Magnitude and without Direction

In conventional vector analysis a vector is defined as a "physical entity that has magnitude and direction." That a vector has a "law of transformation" when the reference frame changes is tacitly assumed as self-evident without any further statement. This obviousness of a law of transformation is due to the simplicity of a Euclidean space and the ease of visualization of entities in such a space. Also in a plane or in a three-dimensional Euclidean space with orthogonal reference frame the inverse of **C** (or rather $\mathbf{C}_t^{-1}$) is identical with **C**, there is no difference between a covariant and a contravariant vector, and the law of transformation loses its importance as a yardstick to recognize physical entities.

With the introduction of generalized coordinates (in electrical and mechanical network and machine studies), the space in which the vectors lie and the reference frame in which they are measured get more complicated, hence the emphasis in the definition of a physical entity (such as a vector) must be shifted to its law of transformation. In tensor analysis a vector is defined as a "physical entity whose law of transformation requires either **C** or $\mathbf{C}^{-1}$ only once." A vector does not necessarily have to possess a *magnitude* or a *direction*. It only has *components* and a definite *law of transformation.*

To define the *magnitude* of a vector it is necessary to introduce the concept of a metric tensor $a_{\alpha\beta}$; and to define *direction* it is necessary to introduce the concept of affine connection, $\Gamma^{\gamma}_{\alpha\beta}$. In their absence there still exist vectors, reference frames, spaces, and other attributes of physical problems; only the two concepts *magnitude* and *direction* are missing.

The Metric Tensor $a_{\alpha\beta}$ *

(*a*) If the self and mutual inductances (and moment of inertia) $a_{\alpha\beta}$ of a synchronous or induction machine are known, their performance can be predetermined under all conditions of operations. For

* *T.A.N.*, Chapter XVIII.

that reason the metric tensor $a_{\alpha\beta}$ plays an all-important part in the study of rotating machinery and of course in tensor analysis. In dynamical problems $a_{\alpha\beta}$ contains all the moments of inertia and products of inertia of the system; in geometrical problems $a_{\alpha\beta}$ contains the direction cosines of the reference axes.

When the *components* of a vector, say i^α (that is, the currents flowing in the various windings), are known, the *magnitude* of that vector is still unknown. In fact, no definition has been given hitherto of what the *resultant* vector **i** represents physically.

(*b*) The absolute magnitude of $|i|$ of a vector i^α is defined in tensor analysis with the aid of the metric tensor as

$$(\text{Magnitude of } i^\alpha)^2 = |i|^2 = a_{\alpha\beta} i^\alpha i^\beta \qquad 32.1$$

Or, if the vector is covariant, like φ_α, its magnitude is defined as

$$(\text{Magnitude of } \varphi_\alpha)^2 = |\varphi|^2 = a^{\alpha\beta}\varphi_\alpha\varphi_\beta \qquad 32.2$$

where $a^{\beta\alpha}$ is the inverse of $a_{\alpha\beta}$. (Physically $a^{\alpha\beta}$ represents short-circuit inductances.)

Since the stored magnetic (kinetic) energy of a system is $T = (\frac{1}{2}) a_{\alpha\beta} i^\alpha i^\beta$, *the magnitude of the current vector i^α at any instant is equal to the square root of twice the magnetic energy stored in the system.*

Raising and Lowering Indices *

Multiplication with the metric tensor $a_{\alpha\beta}$ lowers an upper index, and multiplication with $a^{\alpha\beta}$ raises a lower index, as

$$\begin{gathered} i^\alpha a_{\alpha\beta} = i_\beta \quad \text{or} \quad \varphi_\alpha a^{\alpha\beta} = \varphi^\beta \\ S_{\alpha\beta,\gamma} a^{\gamma\delta} = S_{\alpha\beta}^{\cdot\cdot\delta} \quad \text{and} \quad S_{\alpha\cdot\cdot}^{\cdot\beta\gamma} a_{\beta\delta} = S_{\alpha\delta}^{\cdot\cdot\gamma} \end{gathered} \qquad 32.3$$

Only the indices of *tensors* can be raised or lowered. Exceptions are $\Gamma_{\alpha\beta,\gamma}$ and $[\alpha\beta,\gamma]$, whose *third* indices may be raised or lowered as

$$\Gamma_{\alpha\beta,\gamma} a^{\gamma\delta} = \Gamma_{\alpha\beta}^\delta \quad \text{and} \quad [\alpha\beta,\gamma] a^{\gamma\delta} = \left\{ {\delta \atop \alpha\beta} \right\} \qquad 32.4$$

When an index of a tensor is raised or lowered, its physical meaning also changes. For instance, i_α is *identical* with φ_α (since $\mathbf{L}\cdot\mathbf{i} = \varphi$); similarly $\varphi^\alpha \equiv i^\alpha$. $R_\alpha^{\cdot\beta}$ contains "decrement factors" r/L, $G_\alpha^{\cdot\beta}$ becomes identical with the "rotation tensor" γ_α^β.

* *A.T.E.M.*, Part XV, p. 145.

Covariant (or Absolute) Differentiation

(*a*) Considering the equation of motion 31.13

$$p_\alpha = r_{\alpha\beta}\dot{x}^\beta + a_{\alpha\beta}\frac{d\dot{x}^\beta}{dt} + \Gamma_{\beta\gamma,\alpha}\dot{x}^\beta\dot{x}^\gamma \qquad 32.5$$

the term $a_{\alpha\beta}\dfrac{d\dot{x}^\beta}{dt}$ is not a tensor (since $d\dot{x}^\beta$ is not a tensor, as shown in equation 30.3). Similarly the last term is not a tensor since $\Gamma_{\beta\gamma,\alpha}$ is not (it contains V). However, *the sum of the last two terms, namely,*

$$a_{\alpha\beta}\frac{d\dot{x}^\beta}{dt} + \Gamma_{\beta\gamma,\alpha}\dot{x}^\beta\dot{x}^\gamma = A_\alpha \qquad 32.6$$

is a tensor (since each of the other two terms of the equation is a tensor). That is, the induced voltages do not form a vector (a tensor of valence 1); neither do the generated voltages. But their sum is a tensor, no matter what reference frame is used.

(*b*) This relation is used to define one of the basic operations of tensor analysis that always produces automatically a tensor out of another tensor in spite of the presence of differentiation.

The "covariant (or absolute) derivative" of a vector A^α is defined as

$$\frac{\delta A^\alpha}{dt} = \frac{dA^\alpha}{dt} + \Gamma^\alpha_{\gamma\beta}\, dA^\gamma \frac{dx^\beta}{dt} \qquad 32.7$$

The covariant derivatives of tensors of various valence is defined with the aid of as many $\Gamma^\alpha_{\beta\gamma}$ as the number of valence, e.g.,

$$\frac{\delta A^{\alpha\beta}}{dt} = \frac{dA^{\alpha\beta}}{dt} + \Gamma^\alpha_{\gamma\delta} A^{\gamma\beta}\frac{dx^\delta}{dt} + \Gamma^\beta_{\gamma\delta} A^{\alpha\gamma}\frac{dx^\delta}{dt} \qquad 32.8$$

Covariant derivatives may be defined with respect to tensors of any valence. For instance, in field problems

$$\frac{\delta A^\alpha}{\partial x^\beta} = \frac{\partial A^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\gamma\beta}\, dA^\gamma \qquad 32.9$$

(*c*) The importance of covariant derivatives is that they obey the rules of ordinary derivatives. E.g.,

$$\delta(A_{\alpha\beta}B^{\beta\gamma}) = (\delta A_{\alpha\beta})B^{\beta\gamma} + A_{\alpha\beta}\delta B^{\beta\gamma} \qquad 32.10$$

Hence *in many analyses the presence of $\Gamma^\gamma_{\alpha\beta}$ may be dispensed with* and the analysis performed without being encumbered by $\Gamma_{\alpha\beta,\gamma}$. However, all differentiation symbols then represent covariant differentiations.

The Third Generalization Postulate

(*a*) The preliminary postulate extends the use of a particular arithmetic equation to a large number of analogous cases by replacing each number with an algebraic symbol. The first postulate allows the extension of an equation from one degree (or a few degrees) of freedom to n degrees by replacing each algebraic symbol by an appropriate n-way matrix. The second postulate extends the use of the matric equation (or equations) of a particular system for a large number of systems possessing *the same types of reference frames* by replacing each n-way matrix by an appropriate geometric object.

(*b*) The next step in the generalization of the meaning of symbols concerns reference frames that have more complicated structures. It is comparatively easy to establish, say, the equation of motion of a particle moving on the *plane*. The question arises whether the simple equation of a plane may be generalized to apply to the motion of a particle on a *curved* surface, say on an ellipsoid.

The third generalization postulate states: *An invariant equation, valid for an infinite number of physical systems all possessing a simple type of reference frame, may be generalized to include reference frames of more complicated types, by replacing each geometric object by an appropriate tensor. In particular all ordinary derivatives in the equation are replaced by covariant (or absolute) derivatives.*

(*c*) For instance, the invariant equation

$$e_\alpha = L_{\alpha\beta}\frac{di^\beta}{dt} \qquad 32.11$$

valid for all possible *stationary* networks possessing magnetic (kinetic) energy, is valid for all *rotating* machinery if di^β/dt is replaced by $\delta i^\beta/dt$. That is, the equation of performance of all rotating machines is

$$e_\alpha = L_{\alpha\beta}\frac{\delta i^\beta}{dt} = L_{\alpha\beta}\frac{di^\beta}{dt} + \Gamma_{\beta\gamma,\alpha}i^\beta i^\gamma \qquad 32.12$$

As another example, Newton's law $f = m d\dot{x}/dt$ assumes in a rectilinear reference frame with n degrees of freedom the form

$$f_\alpha = a_{\alpha\beta}\frac{d\dot{x}^\beta}{dt} \qquad 32.13$$

In any curvilinear reference frame and with generalized coordinates, the equation becomes, by virtue of the third postulate,

$$f_\alpha = a_{\alpha\beta}\frac{\delta\dot{x}^\beta}{dt} = a_{\alpha\beta}\frac{d\dot{x}^\beta}{dt} + \Gamma_{\beta\gamma,\alpha}\dot{x}^\beta\dot{x}^\gamma \qquad 32.14$$

where the value of $\Gamma_{\beta\gamma,\alpha}$ depends on the particular reference frame assumed. In holonomic reference frames $\Gamma_{\beta\gamma,\alpha}$ is the Christoffel symbol $[\alpha\beta,\gamma]$ depending only on $a_{\alpha\beta}$; in non-holonomic frames $\Gamma_{\beta\gamma,\alpha}$ assumes a more general form.

The Generalization of Maxwell's Field Equations

As one other illustration of the third generalization postulate, let the field equations of Maxwell, given in equation 14.7, be considered. The equations are valid in rectilinear reference frames that may move with a *uniform* velocity along a straight line. If the axes are curvilinear and the reference frame has an accelerated motion, then equation 14.7 assumes the form

$$\begin{array}{lll|l}
\text{I} & \dfrac{\delta H^{\alpha\beta}}{\partial x^\beta} = s^\alpha & & \dfrac{\partial H^{\alpha\beta}}{\partial x^\beta} + \Gamma^\alpha_{\gamma\beta} H^{\gamma\beta} + \Gamma^\beta_{\gamma\beta} H^{\alpha\gamma} = s^\alpha \\
\text{II} & \dfrac{\delta F^{\alpha\beta}}{\partial x^\beta} = 0 & & \dfrac{\partial F^{\alpha\beta}}{\partial x^\beta} + \Gamma^\alpha_{\gamma\beta} F^{\gamma\beta} + \Gamma^\beta_{\gamma\beta} F^{\alpha\gamma} = 0 \\
\text{III} & \dfrac{\delta s^\alpha}{\partial x^\alpha} = 0 & & \dfrac{\partial s^\alpha}{\partial x^\alpha} + \Gamma^\alpha_{\beta\alpha} s^\beta = 0 \\
\text{IV} & F_{\alpha\beta} = \dfrac{\delta \varphi_\alpha}{\partial x^\beta} - \dfrac{\delta \varphi_\beta}{\partial x^\alpha} & & F_{\alpha\beta} = \dfrac{\partial \varphi_\alpha}{\partial x^\beta} - \dfrac{\partial \varphi_\beta}{\partial x^\alpha} - \Gamma^\gamma_{\alpha\beta}\varphi_\gamma + \Gamma^\gamma_{\beta\alpha}\varphi_\gamma
\end{array} \qquad 32.15$$

where the covariant derivatives are defined in equations 32.7 and 32.8. (A detailed analysis is given in another publication.*) *These are the forms of the Maxwellian equations that apply to rotating electrical machinery.*

Again $\Gamma_{\alpha\beta,\gamma}$ implied in the covariant derivatives depends on the reference frame used. In the special case when $\Gamma_{\alpha\beta,\gamma} = [\alpha\beta,\gamma]$ (that occurs in most field problems but not in rotating machinery), Maxwell's equations assume the very simple form

$$\begin{array}{ll|ll}
\dfrac{1}{\sqrt{-a}} \dfrac{\partial \sqrt{-a}\, H^{\alpha\beta}}{\partial x^\beta} = s^\alpha & \text{I} & \dfrac{\partial \sqrt{-a}\, s^\alpha}{\partial x^\beta} = 0 & \text{III} \\
\dfrac{\partial \sqrt{-a}\, F^{\alpha\beta}}{\partial x^\beta} = 0 & \text{II} & F_{\alpha\beta} = \dfrac{\partial \varphi_\alpha}{\partial x^\beta} - \dfrac{\partial \varphi_\beta}{\partial x^\alpha} & \text{IV}
\end{array} \qquad 32.16$$

They are practically the same as equation 14.7 except that the determinant a of the metric tensor $a_{\alpha\beta}$ also appears in the equations as a scalar multiplier.

* Kron, "Invariant Form of the Maxwell-Lorentz Field Equations for Accelerated Systems," *Journal of Applied Physics*, March, 1938, p. 196.

The Expansion of a Tensor Equation

Again it is emphasized that the use of the third postulate simplifies the problem only during the analysis. Before the constants of a particular engineering structure may be put into the equations, the tensor equations have to be expanded. In particular:

1. The tensors (like δi^{α}) have to be replaced by their equivalent geometric objects.
2. The geometric objects have to be replaced by the n-matrices of the particular reference frame under discussion.
3. The n-matrices have to be replaced by algebraic symbols.
4. The algebraic symbols have to be replaced by their numerical value.

That is, the more the analysis is condensed, the more *routine* work remains that has to be performed eventually. Of course, without condensation the analytical work and the visualization of the phenomena would be in many cases either prohibitively complicated or impossible.

The Establishment of Tensor Equations

The main purpose of this book is to establish equations of performance of electrical engineering systems in a rigorous manner. To accomplish that, certain elementary concepts of tensor analysis have been introduced.

The purpose of tensor analysis, however, is not merely to establish equations of performance in a rigorous manner. That is only a secondary role. A far more important role of the tensorial concepts is to establish the performance of physical systems in terms of actually existing, *measurable quantities*, that is, in terms of *tensors* only. In stationary networks this last role is of secondary importance since practically any method gives measurable quantities. But in case of rotating machinery, that is not so. In the familiar steady-state problems long experience has already established certain routine methods that give measurable quantities, but in problems of hunting, little or no such engineering experience exists.

To establish the equations of hunting of dynamical systems in terms of measurable physical quantities (tensors) only, still more advanced concepts of tensor analysis have to be employed; these, however, are not undertaken in this book. One advantage of such an analysis is the possibility of establishing equivalent stationary networks that correspond to the hunting system. If the equation of hunting of a machine is not a tensor equation, it is impossible to establish a stationary network that corresponds term by term to the given non-tensor equa-

tion.* The equations of hunting of slip-ring machines, as given in Chapter 29, are not tensor equations.

In general, an equation of a physical system may be represented by a model (equivalent circuit) only if the equation is a tensor equation.

* Kron, "Equivalent Circuits for the Hunting of Electrical Machinery," *Trans. A.I.E.E.*, 1942.

INDEX